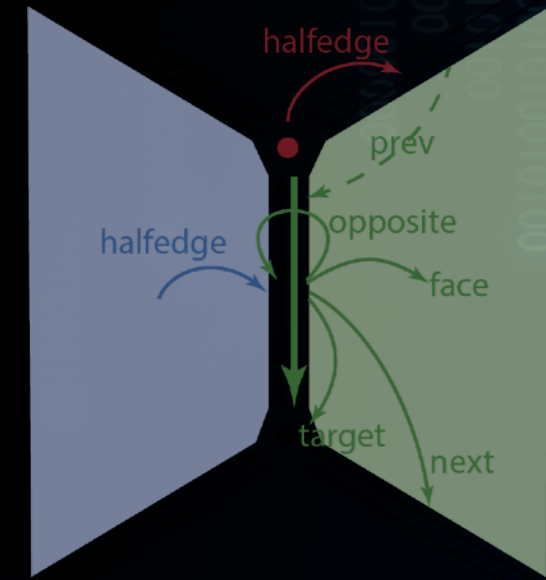


# MODELISATION GEOMETRIQUE

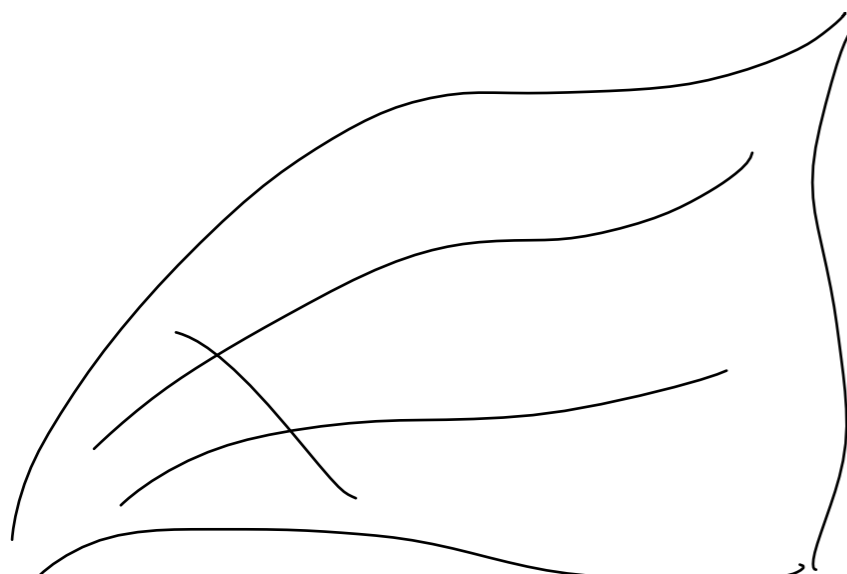


Alexandra Bac

POLYTECH 4A INFORMATIQUE **REVA**

## 3 - GÉOMÉTRIE DES SURFACES

Certaines illustrations sont issues du livre « polygon mesh processing »

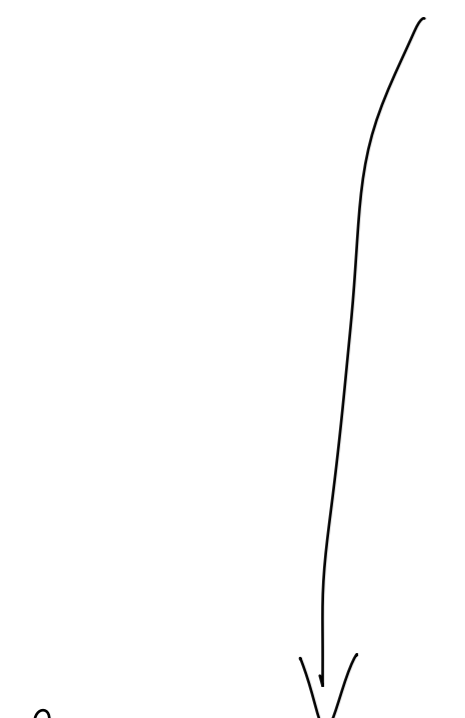


→ mesh

→ parametric

$$f: \frac{\mathbb{R}^2}{\text{param}} \longrightarrow \frac{\mathbb{R}^3}{\text{3D surface}}$$

$$(u, v) \longmapsto f(u, v) \in \mathbb{R}^3$$



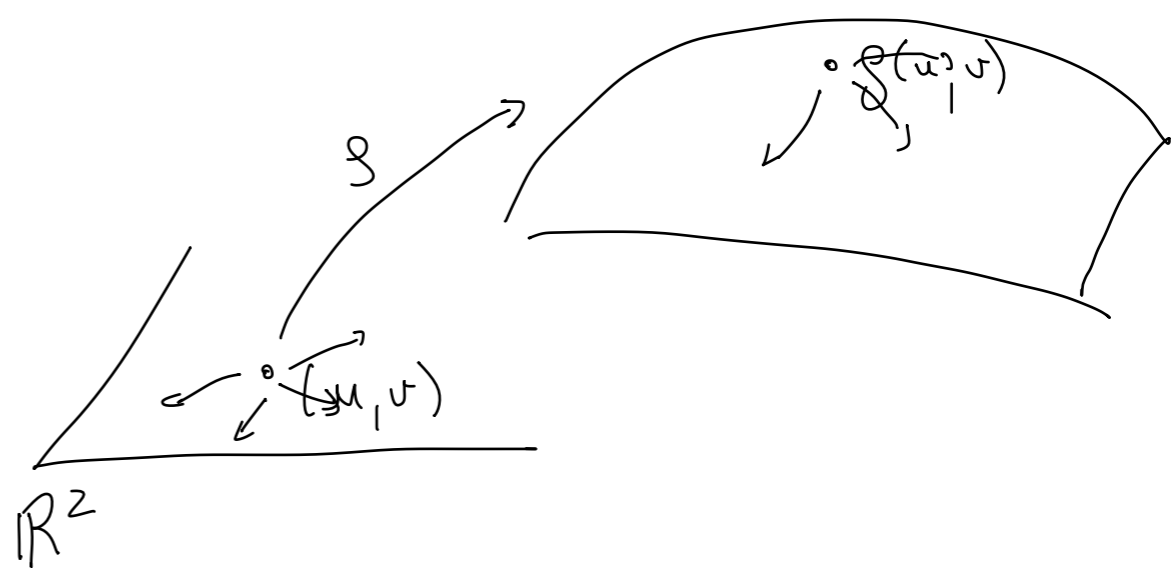
implicit

$$f(x, y, z)$$

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 - r^2 = 0$$

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$S = \{ (x, y, z) ; f(x, y, z) = 0 \}$$



*Chapitre 2*

**MAILLAGES**

*Chapitre 3*

**GÉOMÉTRIE DES  
SURFACES**

*Chapitre 1*

**MODÉLISATION DES  
SURFACES**

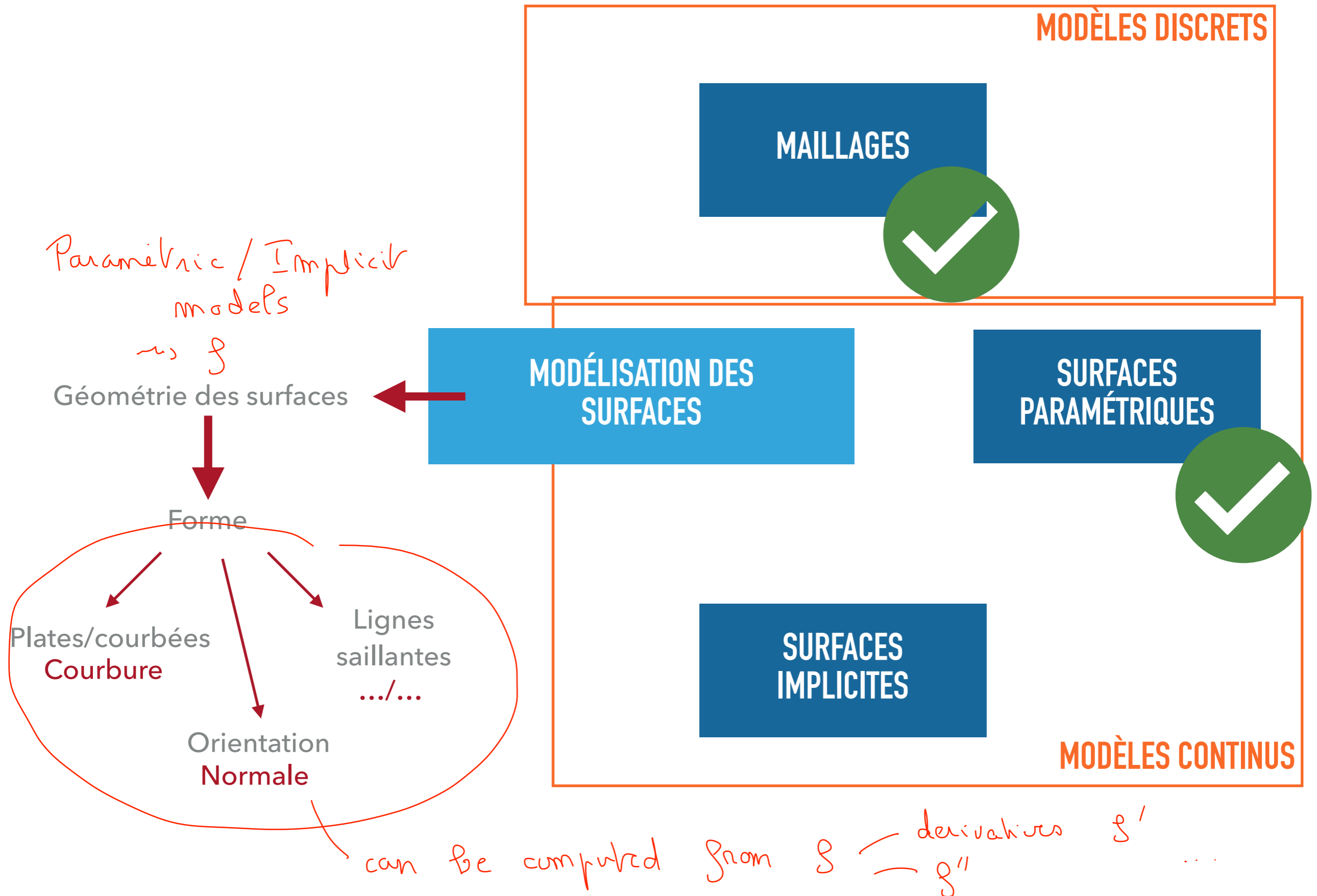
*Chapitre 4 (+ 5A)*

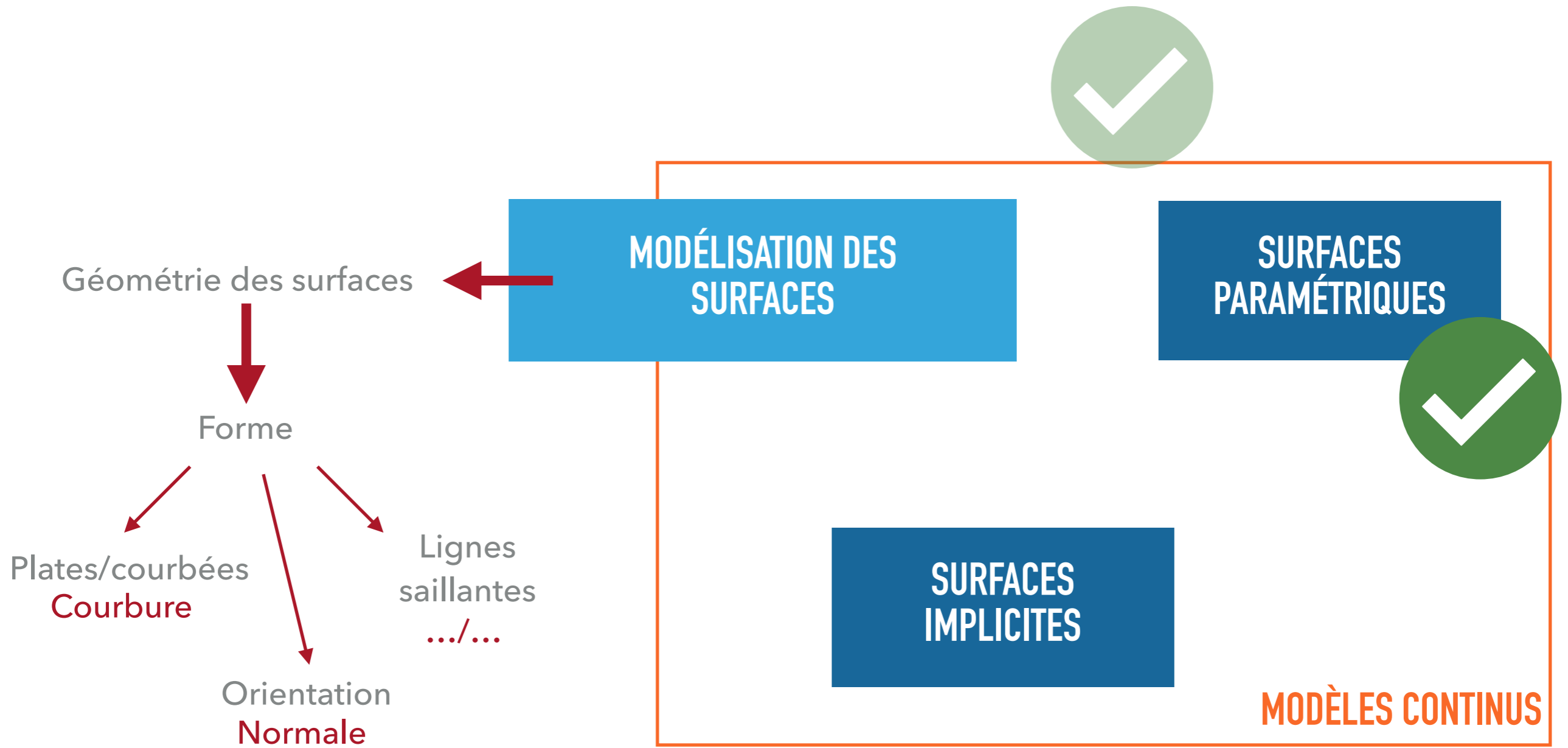
**SURFACES  
PARAMÉTRIQUES**

*Chapitre 5*

**SURFACES  
IMPLICITES**

# « MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES





**POUR L'INTUITION :  
GÉOMÉTRIE DES COURBES  
PARAMÉTRIQUES**

The image shows a close-up of a financial data table. The table consists of several rows of numerical values. The values are arranged in a repeating pattern. The first column contains values: 5.56, 3.24, 9.62, 36, 56, 24, 62, 36, 56, 24. The second column contains values: +740.21, +122.56, +140.04, +180.98, +740.21, +122.56, +140.04, +180.98, +740.21. The third column contains values: -, -, -, -, -, -, -, -, -. The table is partially obscured by a blue overlay on the right side.

5.56	+740.21	-
3.24	+122.56	-
9.62	+140.04	-
36	+180.98	-
56	+740.21	-
24	+122.56	-
62	+140.04	-
36	+180.98	-
56	+740.21	-
24	+122.56	-
62	+140.04	-
36	+180.98	-
56	+740.21	-

# COURBES CARTÉSIENNES

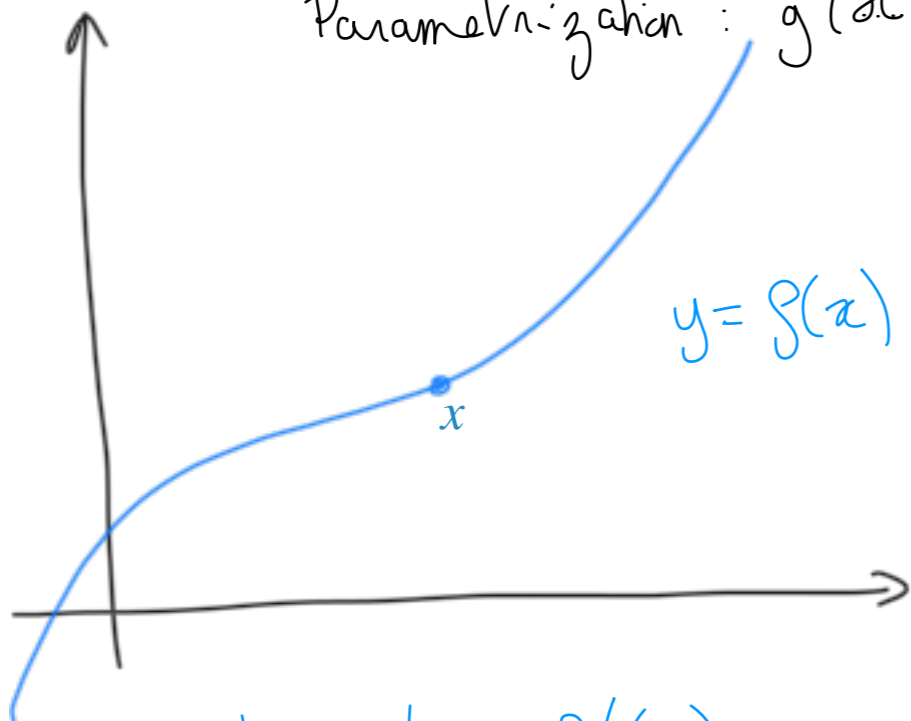
# « MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

Modélisées sous la forme  $y = f(x)$

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

Cartesian curve

Paramétrisation :  $g(x) = \begin{pmatrix} x \\ f(x) \end{pmatrix}$



→ tangente :  $f'(x)$

→ convex  
concave

$$f''(x) \geq 0$$

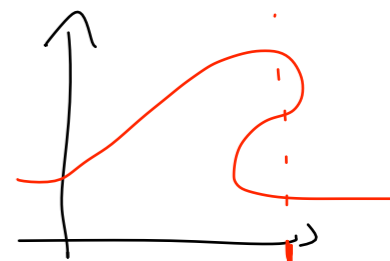
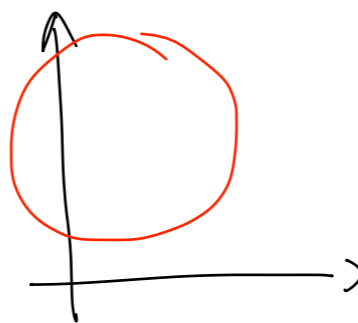
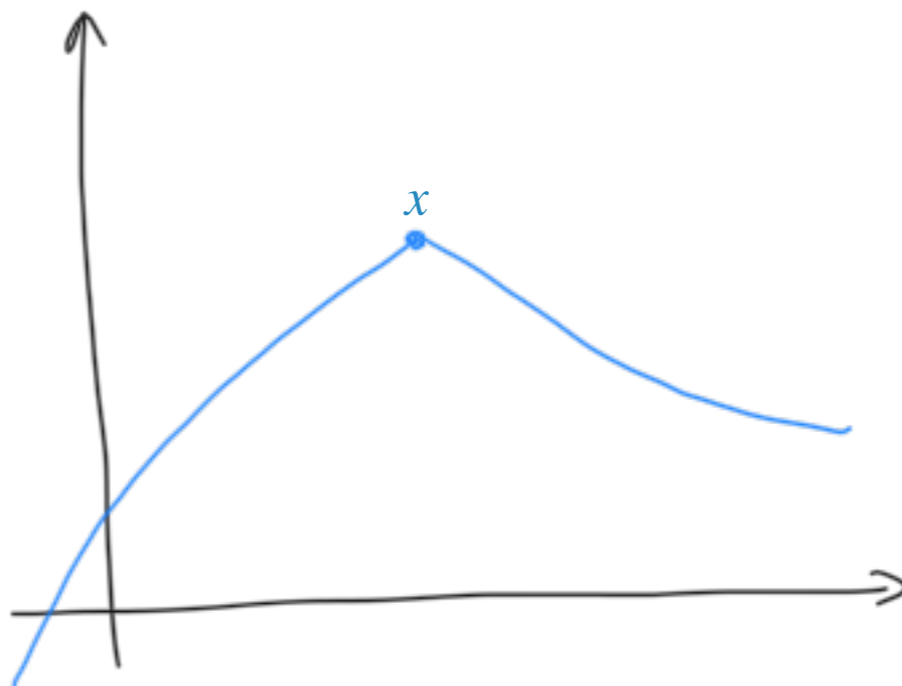
$$\leq 0$$

Paramétr. rep : (curve)

$$g: \mathbb{R} \rightarrow \mathbb{R}^2$$

$$t \mapsto g(t)$$

Connues depuis la terminale ...



cannot model

Cartesian curves  $\subseteq$  parametric

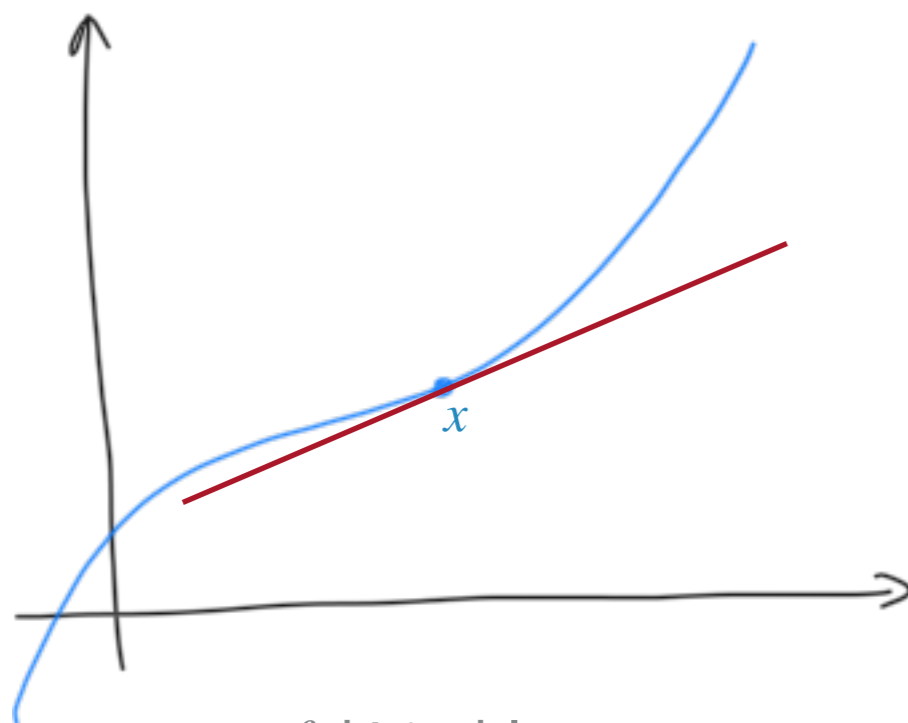


# « MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

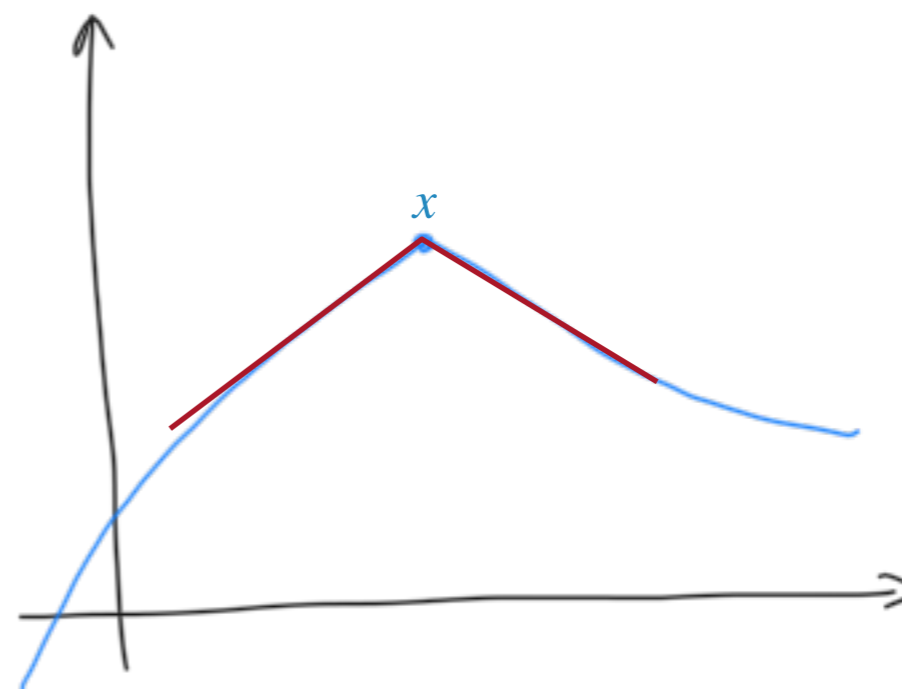
---

Modélisées sous la forme  $y = f(x)$

Connues depuis la terminale ...



$f$  dérivable



$f$  non dérivable

Courbe proche d'une droite de pente  $f'(x)$

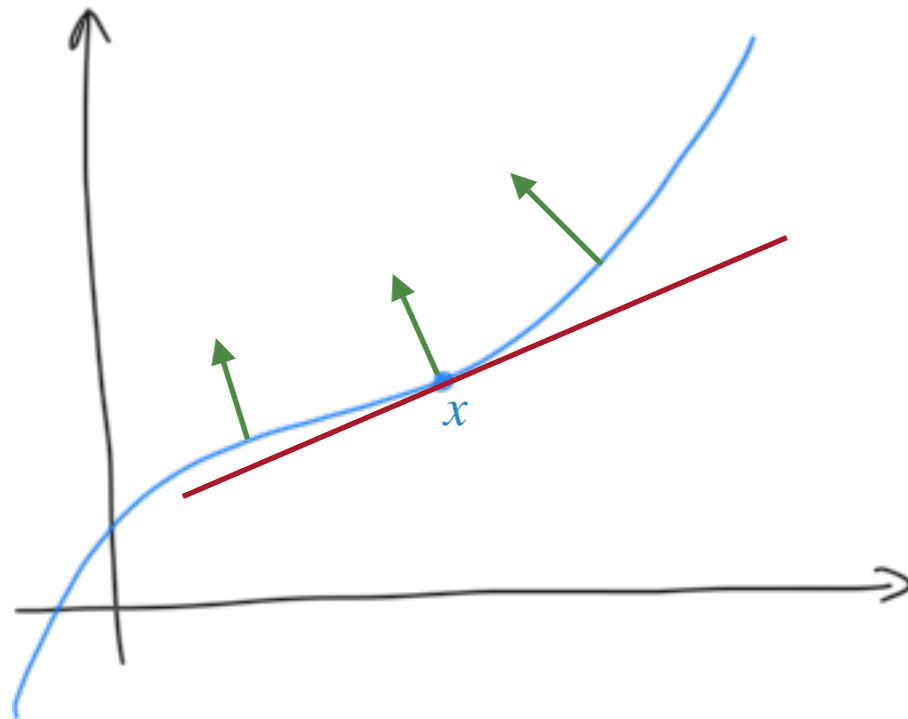
**Tangente**

# « MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

---

Modélisées sous la forme  $y = f(x)$

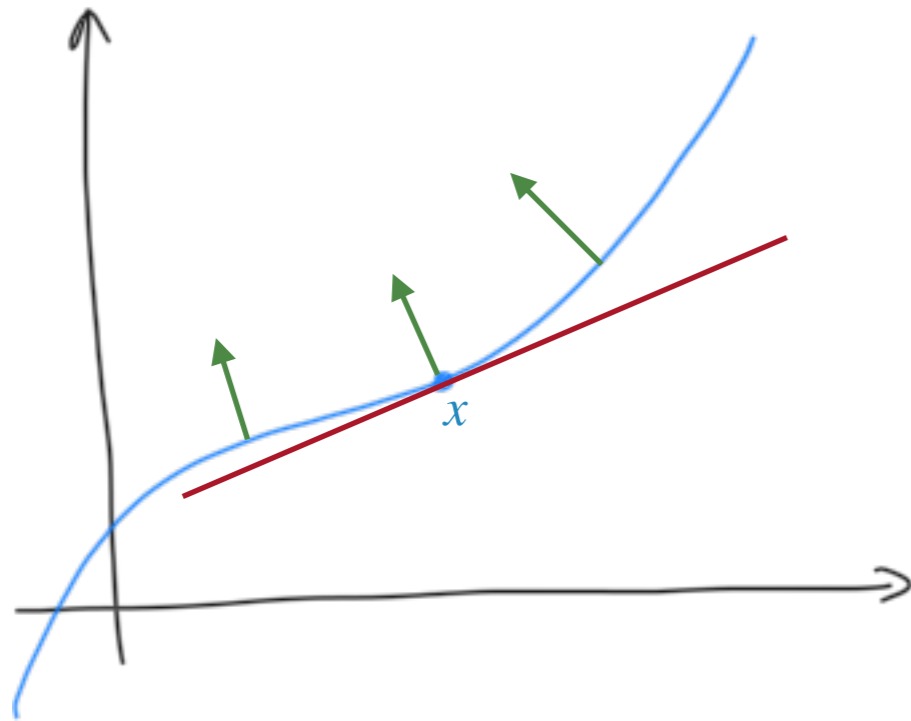
Connues depuis la terminale ...



Tangent )  
Normale :  $\rightarrow g'$   
orthogonale à la courbe/tangente

# « MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

Modélisées sous la forme  $y = f(x)$

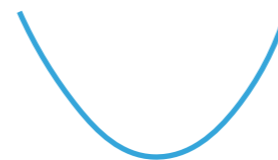


Connues depuis la terminale ...

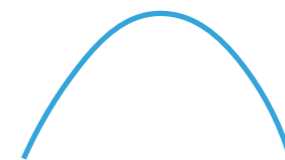
curvature :  
variations of the normal vector  
↳  $f''$

**Dérivées secondes :**

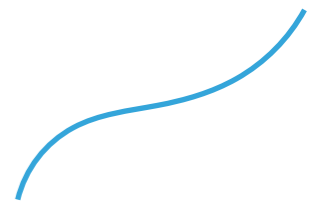
$f''(x)$  indique la concavité de la courbe



Concave  
 $f''(x) > 0$



Convexe  
 $f''(x) < 0$



Inflexion  
 $f''(x) = 0$



**Courbure**

5.56	+740.21	-0
3.24	+122.56	-0
0.62	+140.04	-0
.36	+180.98	-0
.56	+740.21	-0
.24	+122.56	-0
.62	+140.04	-0
.36	+180.98	-0
.56	+740.21	-0

# COURBES PARAMÉTRIQUES GÉNÉRALES

Cf. Chapitre 3

**COURBES**

Courbe finie :

$$f : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}^2 \text{ (ou } \mathbb{R}^3)$$

$$g : \mathbb{R} \rightarrow \mathbb{R}^2 \text{ (ou } \mathbb{R}^3) \dots$$

$$g' \text{ ?}$$



$f(t) \in \mathbb{R}^2/\mathbb{R}^3$  donc :

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ [f_3(t)] \end{pmatrix}$$

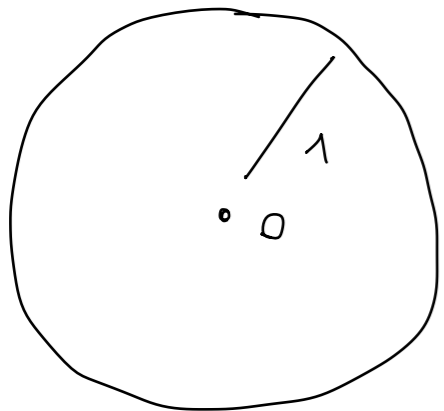
$g_1 : \mathbb{R} \rightarrow \mathbb{R}$   
(1st coord)

$g_2 : \mathbb{R} \rightarrow \mathbb{R}$   
(2nd coord)

$$f_i : \mathbb{R} \rightarrow \mathbb{R}$$

Tangente	Courbure
$f'(t)$	$f''(t)$





$\mathcal{S}'$

$$\mathcal{S}: \mathbb{R} \rightarrow \mathbb{R}^2$$
$$\mathcal{S}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$t \in [0, 2\pi]$$

$$= \begin{pmatrix} \mathcal{S}_1(t) = \cos t \\ \mathcal{S}_2(t) = \sin t \end{pmatrix}$$

$$\mathcal{S}_1, \mathcal{S}_2: \mathbb{R} \rightarrow \mathbb{R}$$

$$\mathcal{S}' = \begin{pmatrix} \mathcal{S}'_1 \\ \mathcal{S}'_2 \end{pmatrix}$$

$$\mathcal{S}': \mathbb{R} \rightarrow \mathbb{R}^2$$

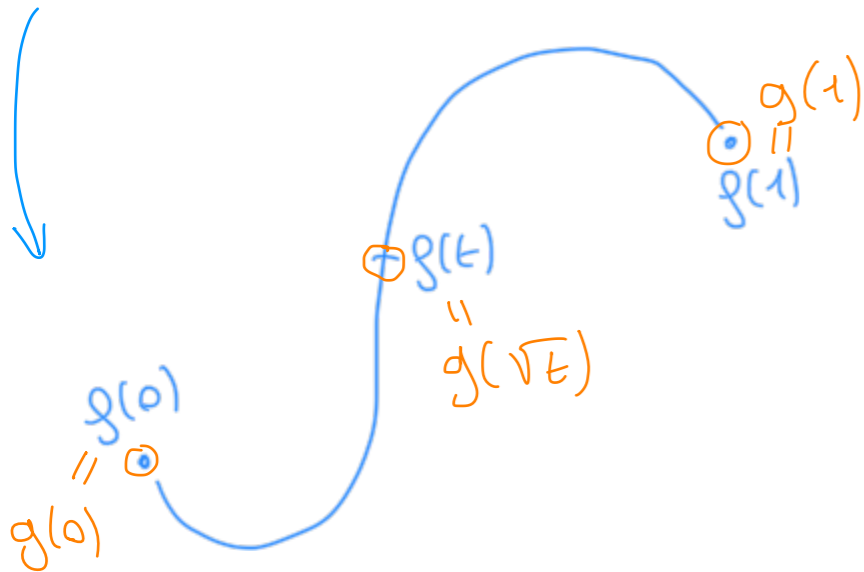
# « MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

$$g: [0,1] \rightarrow \mathbb{R}^2$$

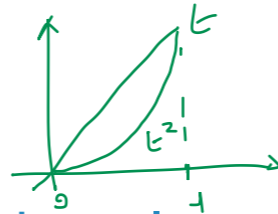
Cf. Chapitre 3

$$g: [0,1] \rightarrow \mathbb{R}^2$$

same as the curve  $g$



Comme vu au chapitre 3 :



décrit la même courbe parcourue à une vitesse différente

$$g(t) = f(t^2)$$

curve  $g$

$t$  ~ time  
 $\sim$  speed along the curve (met unique...)  
 $g'(t) = 2t \times g'(t^2)$

$$g(0) = g(0)$$

$$g(1) = g(1)$$

Normal parametrization:

the one such that the speed along the curve is 1  
 $\forall t \quad \|g'(t)\| = 1$

**Paramétrisation normale**

Tangente

Courbure

$$f'(s)$$

$$f''(s)$$

!

tangent  $g'$

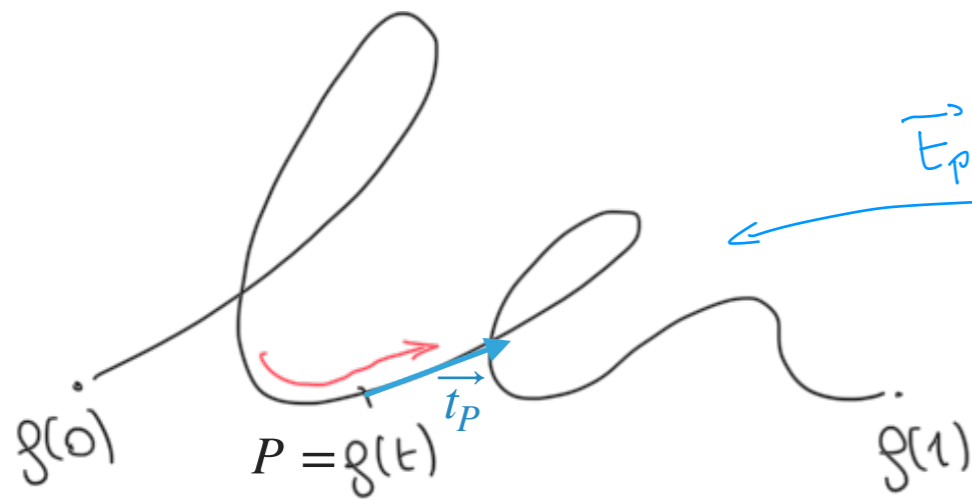
~~curvature  $g''$~~

take the speed into account...

$$g''(t) = 2g'(t^2) + (2t)(2t)g''(t^2) = 2g'(t^2) + 4t^2g''(t^2)$$

very  $\neq$  .....

# TANGENTE / NORMALE



En paramétrisation normale :

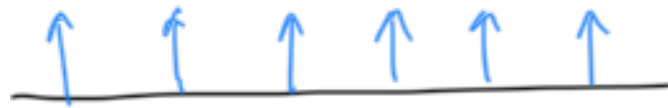
$$\vec{t}_P = f'(s) \quad \text{— } s : \text{time when the param. is normal}$$

En paramétrisation quelconque :

$$\vec{t}_P = \frac{f'(t)}{\|f'(t)\|} \quad \text{— speed}$$



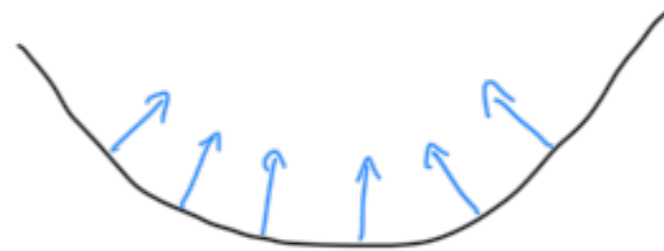
# COURBURE



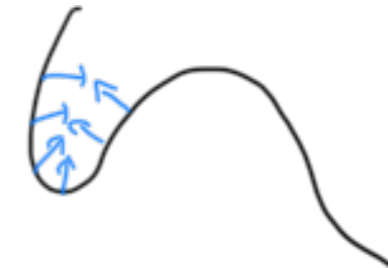
Non courbé



Normale/  
tangente  
constante



Moyennement courbé



Fortement courbé



Variation  
violente de la  
normale/  
tangente

Courbure  $\leftrightarrow$  variation de la normale

Moyennant la paramétrisation ...

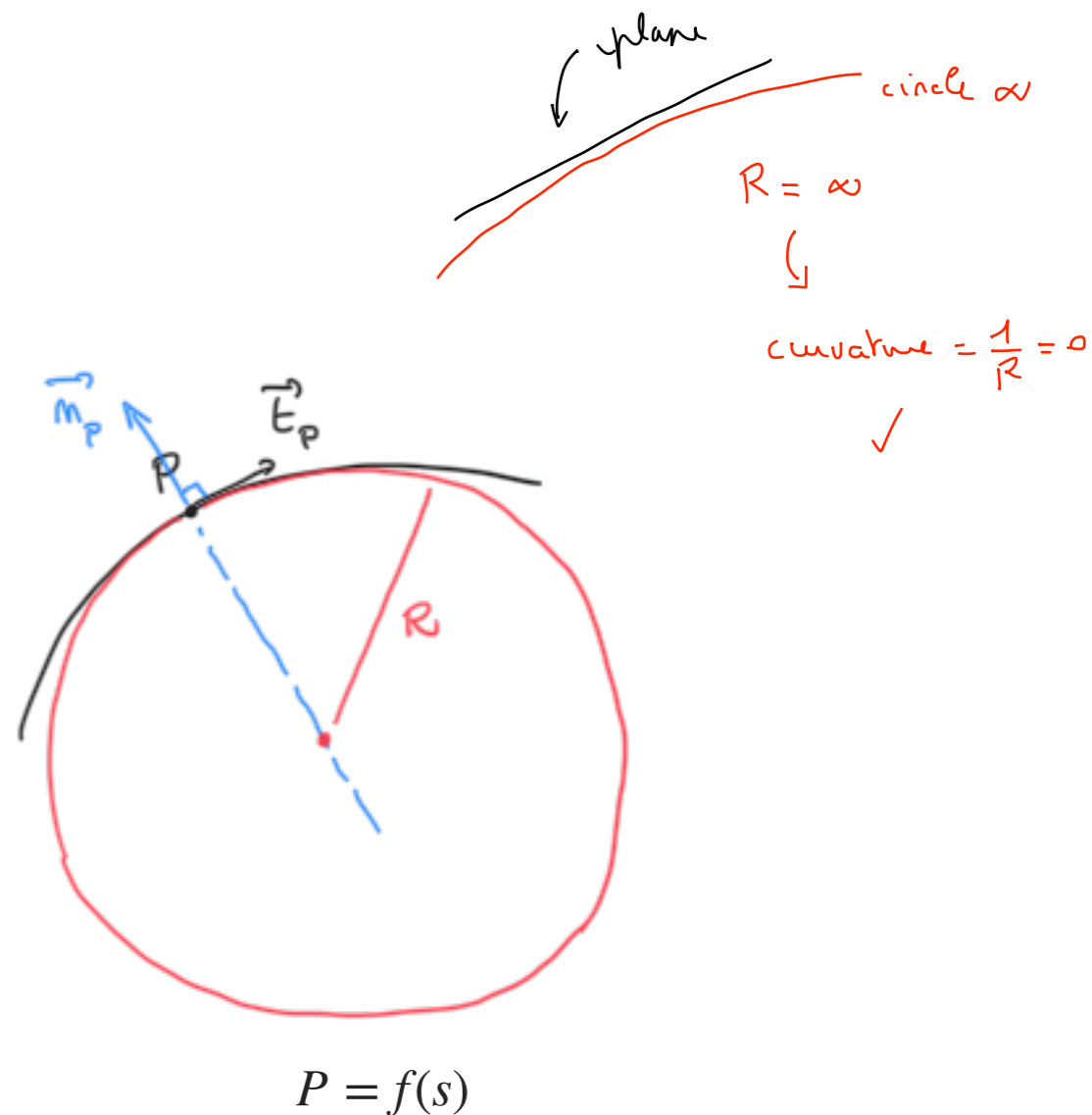
# COURBURE

En paramétrisation normale, on montre que :

$$\frac{d\vec{t}_P}{ds} = \frac{1}{R} \cdot \vec{n}_P = k_P \cdot \vec{n}_P$$

$R$  rayon de courbure

courbure au point  $P$



curvature increases

↳  $R \rightarrow$  small

$$\text{curvature} = \frac{1}{R}$$

↓  
large



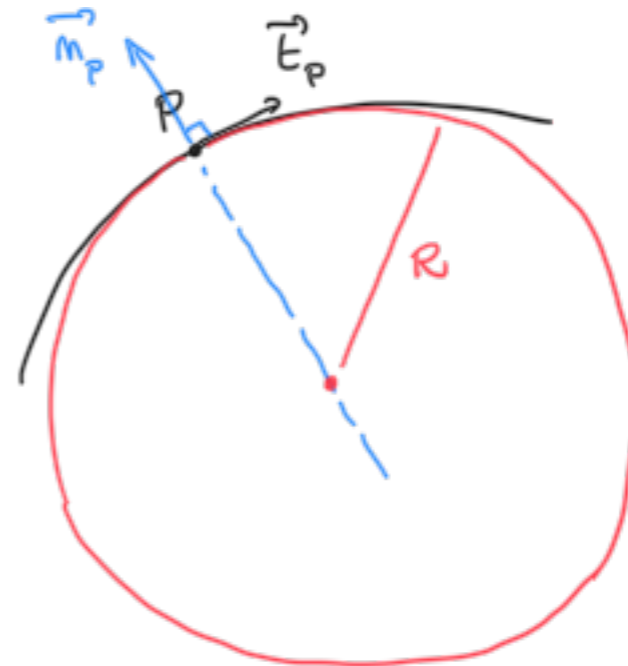
# COURBURE

En paramétrisation normale, on montre que :

$$\frac{d\vec{t}_P}{ds} = \frac{1}{R} \cdot \vec{n}_P = k_P \cdot \vec{n}_P$$

$R$  rayon de courbure

courbure au point  $P$



$$P = f(s)$$

En paramétrisation quelconque :

$$k_P \cdot \vec{n}_P = \frac{f'(t) \wedge f''(t)}{\|f'(t)\|^3}$$

← curvature of a curve of eq.  $\mathcal{S}$

# TRIÈDRE DE FRENET (COURBES GAUCHES)

On pose le vecteur de la binormale :

$$\vec{b}_P = \vec{t}_P \wedge \vec{n}_P$$

$$\frac{d\vec{t}_P}{ds} = \frac{1}{R} \cdot \vec{n}_P = k_P \cdot \vec{n}_P$$

$R$  rayon de courbure

courbure

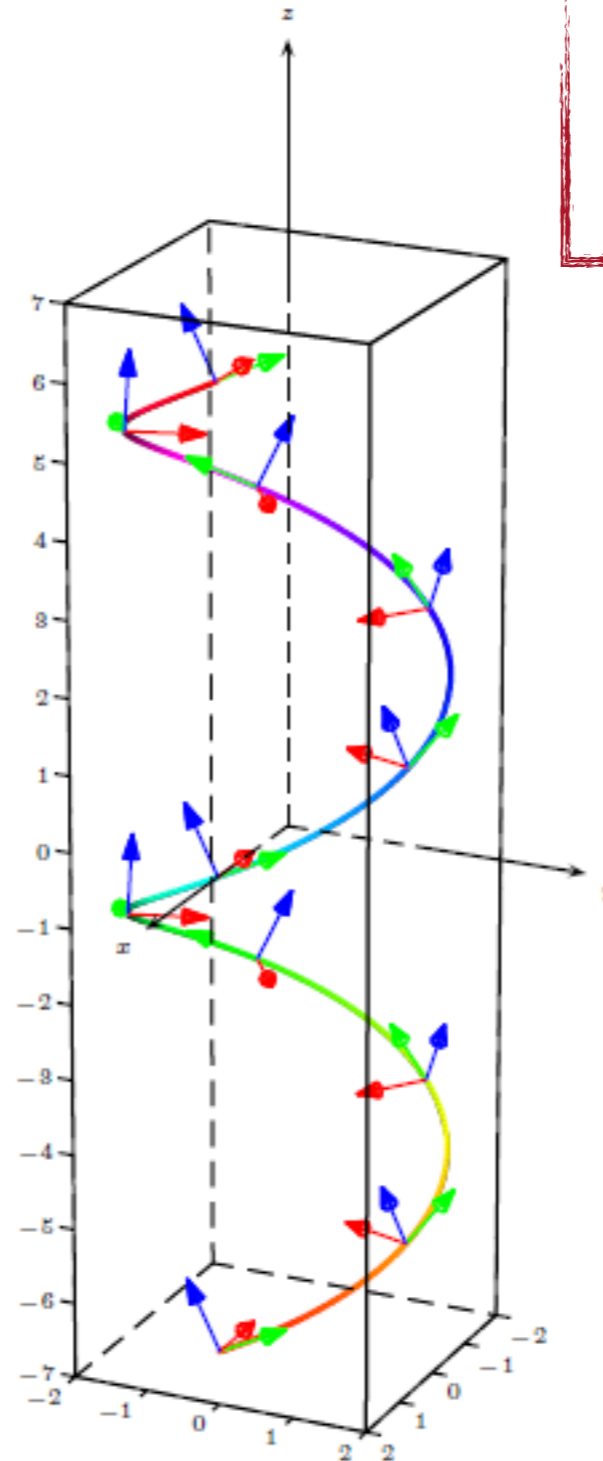
$$\frac{d\vec{b}_P}{ds} = \frac{1}{\tau_P} \cdot \vec{n}_P$$

rayon de torsion

$\frac{1}{\tau_P}$  torsion

Repère de Frénet :

$$(P, \vec{t}_P, \vec{n}_P, \vec{b}_P)$$



5.56	+740.21	-
3.24	+122.56	-
9.62	+140.04	-
.36	+180.98	-
.56	+740.21	-
.24	+122.56	-
.62	+140.04	-
.36	+180.98	-
.56	+740.21	-

# BONUS : FORMES QUADRATIQUES

# FORMES QUADRATIQUES ET MATRICES

Forme quadratique sur  $\mathbb{R}^n$ :

$\Phi$  : polynôme de degré 2 exactement ( $n$  variables)



Codable par une matrice  $\Omega$  de taille  $n \times n$  :

$$\Phi(X) = X^t \cdot \Omega \cdot X$$

$$\phi(x, y, z) = X^t \cdot \Omega \cdot X$$

Sur  $\mathbb{R}^3$  :

$$\Phi(x, y, z) = x^2 - y^2 + 2z^2 - xy + 2yz$$

forme quadratique  
↑

forme quad

unique  
mat. sym

$$\Omega = \begin{matrix} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} x \\ y \\ z \end{matrix} & \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \end{matrix}$$

$$xy = yx$$

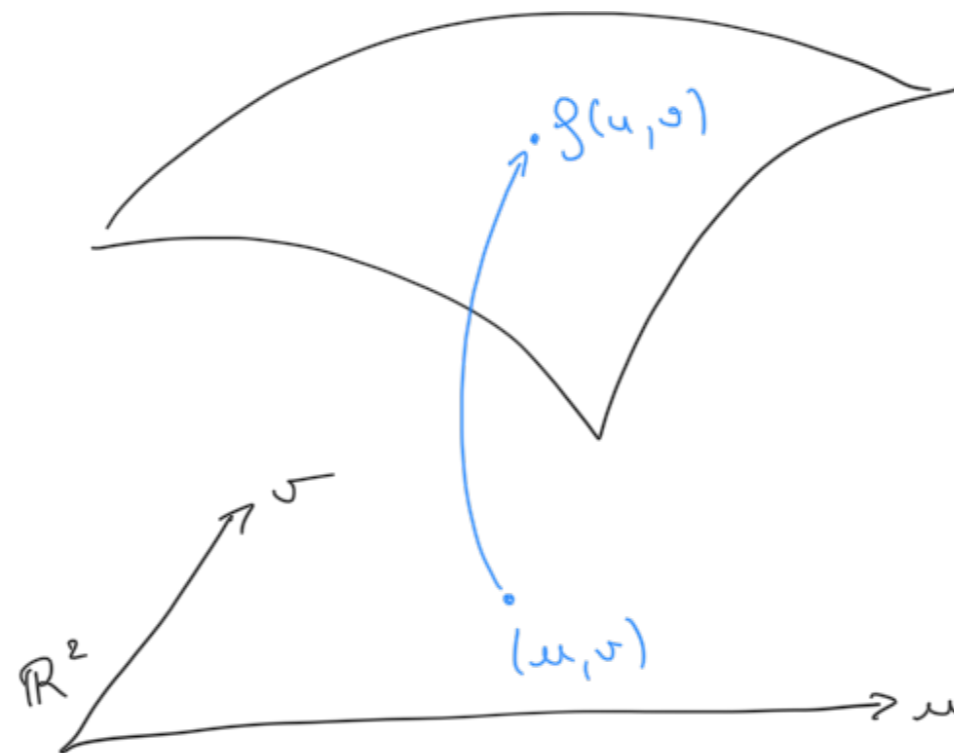
# GÉOMÉTRIE DES SURFACES PARAMÉTRIQUES

# « MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

---

Cadre : surfaces paramétriques

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$
$$(u, v) \mapsto g(u, v) \in \mathbb{R}^3$$



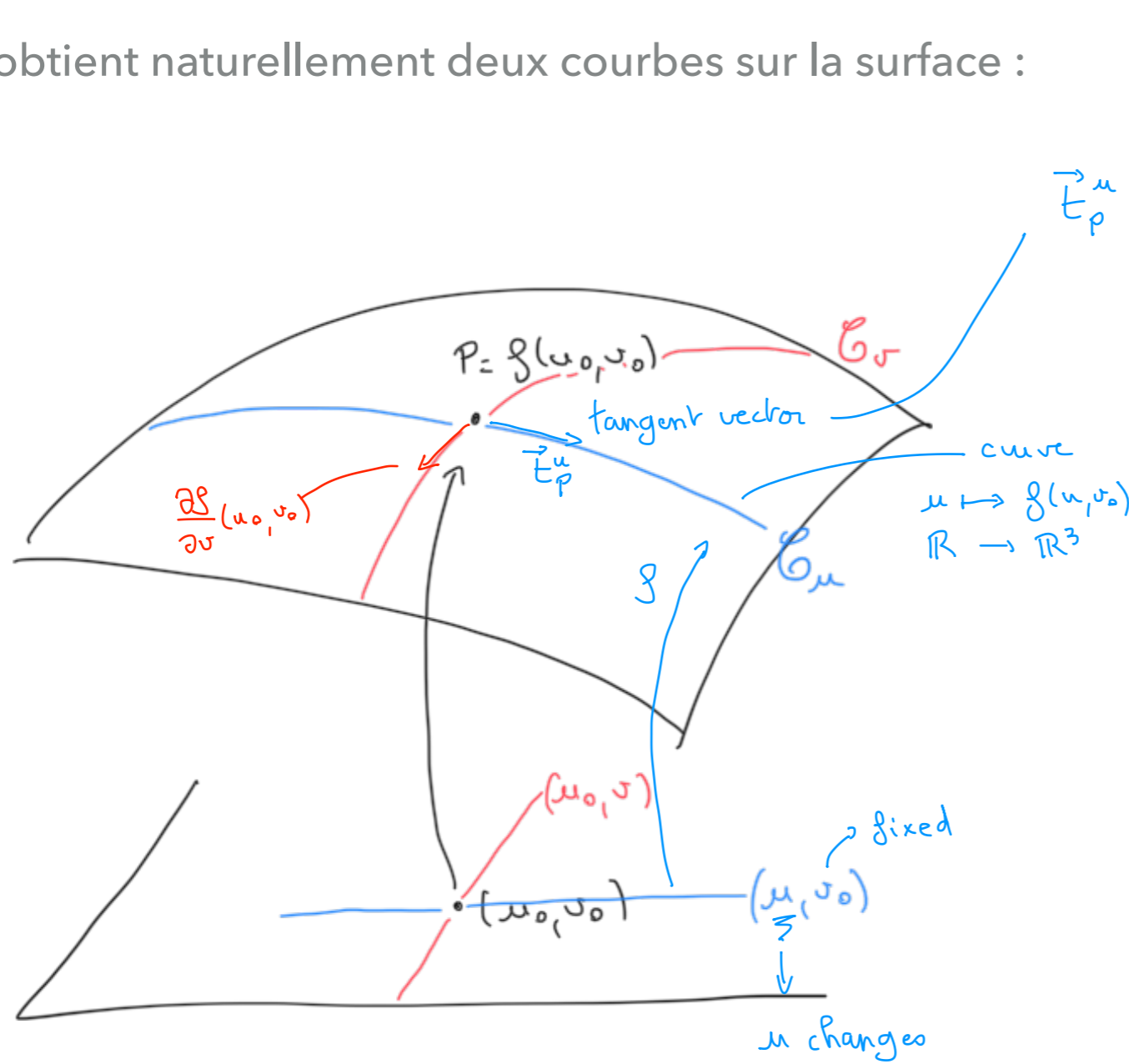


5.56	+740.21	-
3.24	+122.56	-
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.36	+180.98	-
.56	+740.21	-
.24	+122.56	-
.62	+140.04	-
.36	+180.98	-
.56	+740.21	-

**NORMALE**

# « MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

On obtient naturellement deux courbes sur la surface :



$$\vec{t}_p^u = \frac{(g(u, v_0))'}{\|g(u, v_0)'\|}$$

$\frac{\partial g}{\partial u}$

En fixant  $v_0$  :

$$C_u : u \mapsto f(u, v_0)$$

En fixant  $u_0$  :

$$C_v : v \mapsto f(u_0, v)$$



On connaît la tangente de courbes ...

Math...

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

~~\_\_\_\_\_~~  
(u, v)

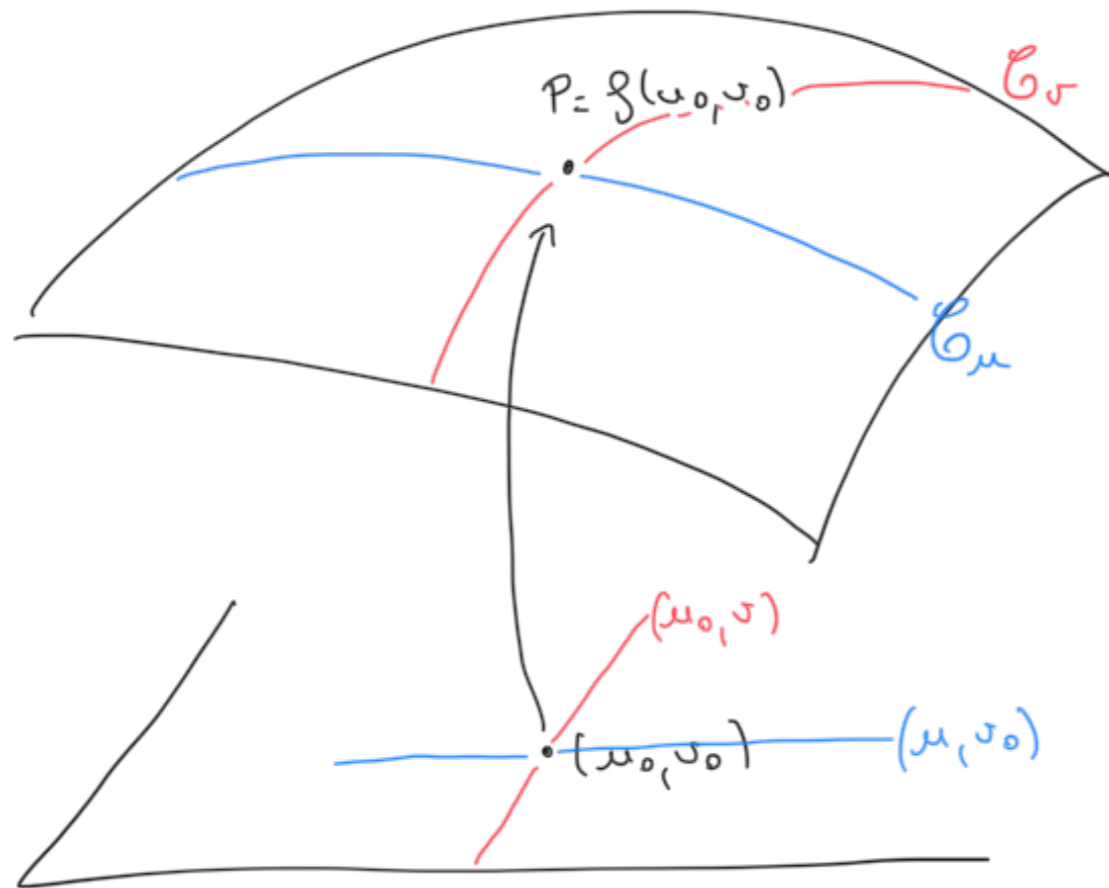
~~S~~

$$\frac{\partial f}{\partial u} \quad / \quad \frac{\partial f}{\partial v}$$

differential  $df$

# « MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

On obtient naturellement deux courbes sur la surface :



En fixant  $v_0$  :

$$C_u : u \mapsto f(u, v_0)$$

En fixant  $u_0$  :

$$C_v : v \mapsto f(u_0, v)$$



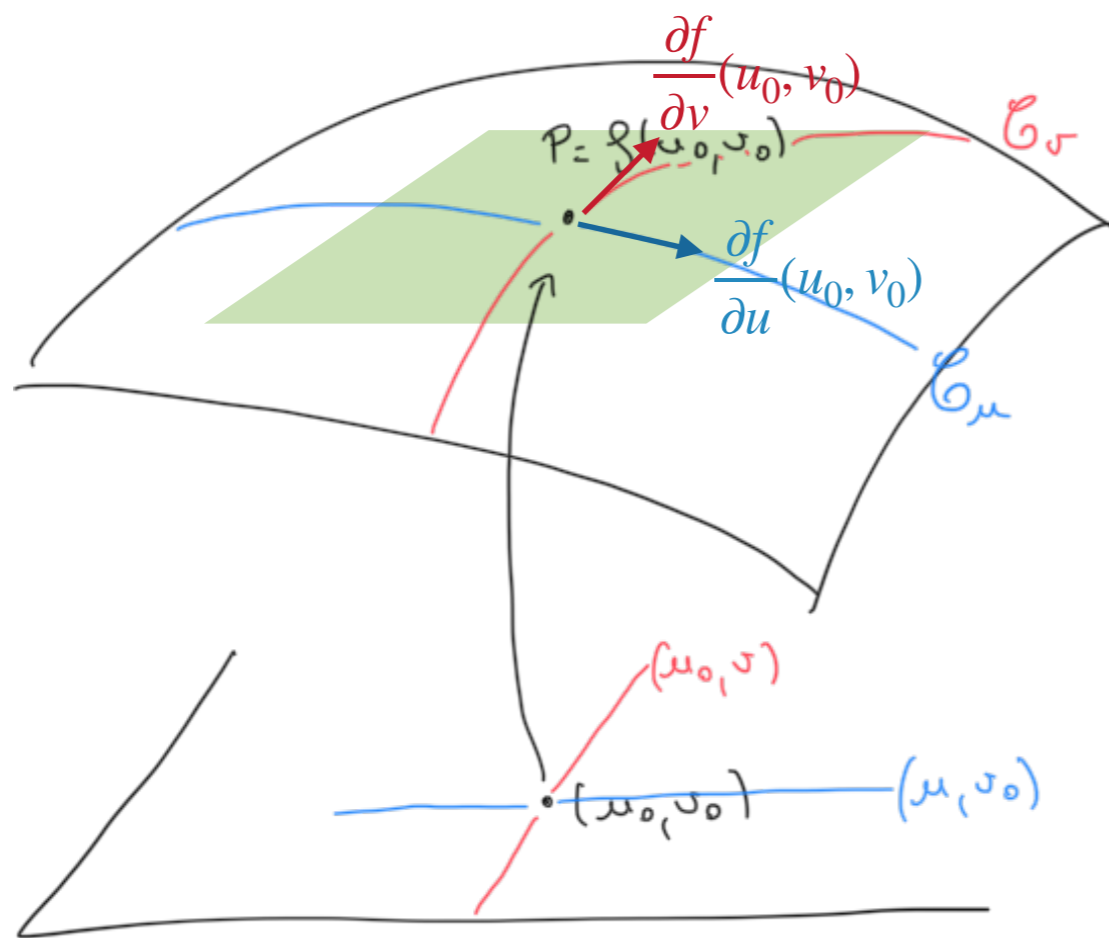
On connaît la tangente de courbes ...

$$\begin{aligned} & (f(u, v_0))' \\ &= \frac{\partial f}{\partial u}(u_0, v_0) \end{aligned}$$

$$\begin{aligned} & (f(u_0, v))' \\ &= \frac{\partial f}{\partial v}(u_0, v_0) \end{aligned}$$

# « MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

On obtient naturellement deux courbes sur la surface :

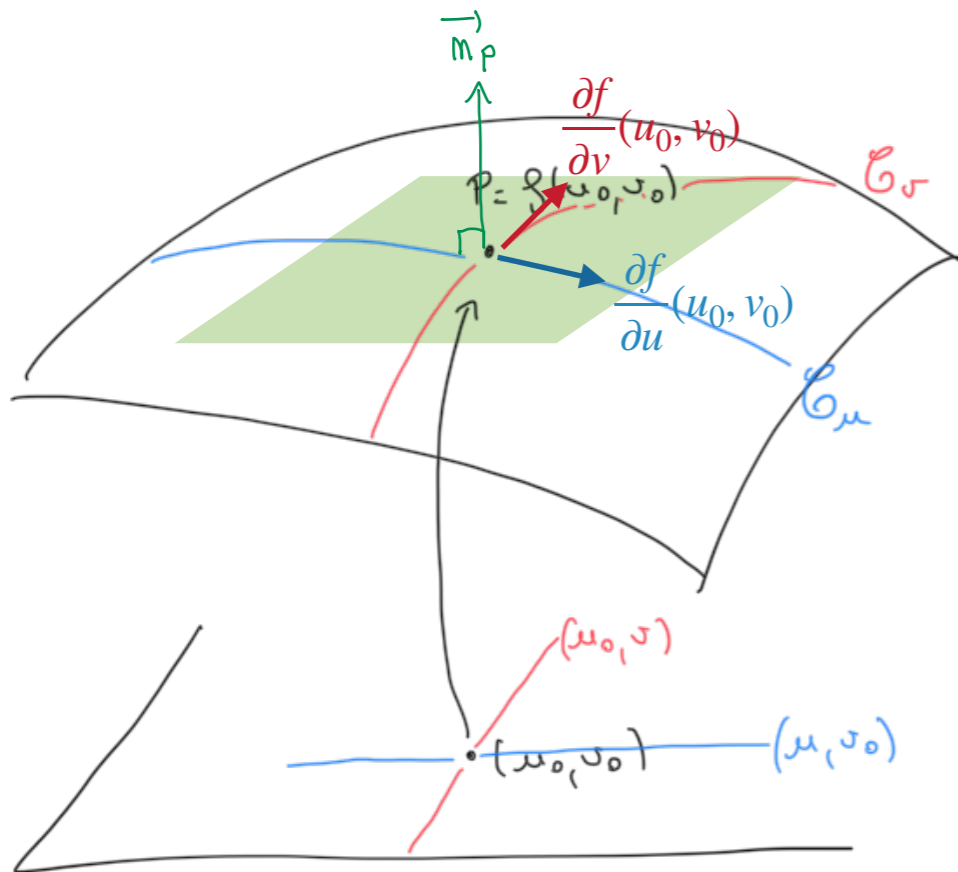


Si  $f$  différentiable en  $(u_0, v_0)$  :

- ▶ Tangentes aux courbes sur la surface forment un plan
- ▶  $\frac{\partial f}{\partial u}(u_0, v_0), \frac{\partial f}{\partial v}(u_0, v_0)$  en forment une base si ...

# « MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

On obtient naturellement deux courbes sur la surface :



$P = f(u_0, v_0)$  **point régulier** si :

- ▶  $\frac{\partial f}{\partial u}(u_0, v_0), \frac{\partial f}{\partial v}(u_0, v_0)$  libres, ie.
- ▶  $\frac{\partial f}{\partial u}(u_0, v_0) \wedge \frac{\partial f}{\partial v}(u_0, v_0) \neq 0$

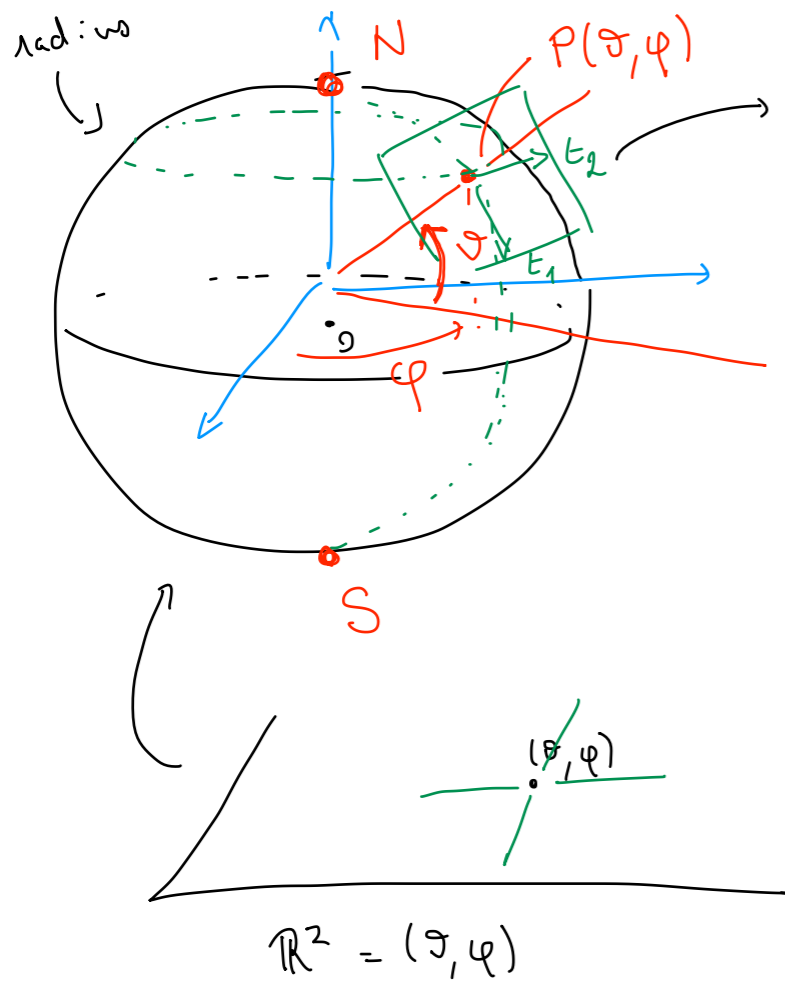
Prop :

si  $P = f(u_0, v_0)$  régulier, le **plan tangent**  $T_P$  en  $P$  a pour repère :

$$\left( P, \frac{\partial f}{\partial u}(u_0, v_0), \frac{\partial f}{\partial v}(u_0, v_0) \right)$$

Son vecteur normal :

$$\overset{\text{unitary}}{\vec{n}_P} = \frac{\frac{\partial f}{\partial u}(u_0, v_0) \wedge \frac{\partial f}{\partial v}(u_0, v_0)}{\left\| \frac{\partial f}{\partial u}(u_0, v_0) \wedge \frac{\partial f}{\partial v}(u_0, v_0) \right\|}$$



$$f(\theta, \varphi) = \begin{pmatrix} R \cos \varphi \cos \theta \\ R \sin \varphi \cos \theta \\ R \sin \theta \end{pmatrix}$$

Tangent vectors:

$$\vec{t}_1 = \frac{\partial f}{\partial \theta} = \begin{pmatrix} -R \cos \varphi \sin \theta \\ -R \sin \varphi \sin \theta \\ R \cos \theta \end{pmatrix}$$

$$\vec{t}_2 = \frac{\partial f}{\partial \varphi} = \begin{pmatrix} -R \sin \varphi \cos \theta \\ R \cos \varphi \cos \theta \\ 0 \end{pmatrix} \leftarrow$$

Are all points regular ???

$$t_1 \wedge t_2 = \begin{pmatrix} -R \cos \varphi \sin \theta & -R \sin \varphi \cos \theta \\ -R \sin \varphi \sin \theta & R \cos \varphi \cos \theta \\ R \cos \theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -R^2 \cos \varphi \cos^2 \theta \\ -R^2 \sin \varphi \cos^2 \theta \\ -R^2 \cos^2 \varphi \sin \theta \cos \theta - R^2 \sin^2 \varphi \cos \theta \sin \theta \end{pmatrix}$$

$$= R^2 \sin \theta \cos \theta (\cos^2 \varphi + \sin^2 \varphi)$$

" 1

$R \neq 0$

$$= \begin{pmatrix} R^2 \cos \varphi \cos^2 \theta \\ R^2 \sin \varphi \cos^2 \theta \\ R^2 \sin \theta \cos \theta \end{pmatrix} \stackrel{?}{=} \vec{0}$$

is  $\theta = \pm \frac{\pi}{2}$

$\downarrow$   
N, S poles  $\leadsto$  not regular

— singular

# « MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

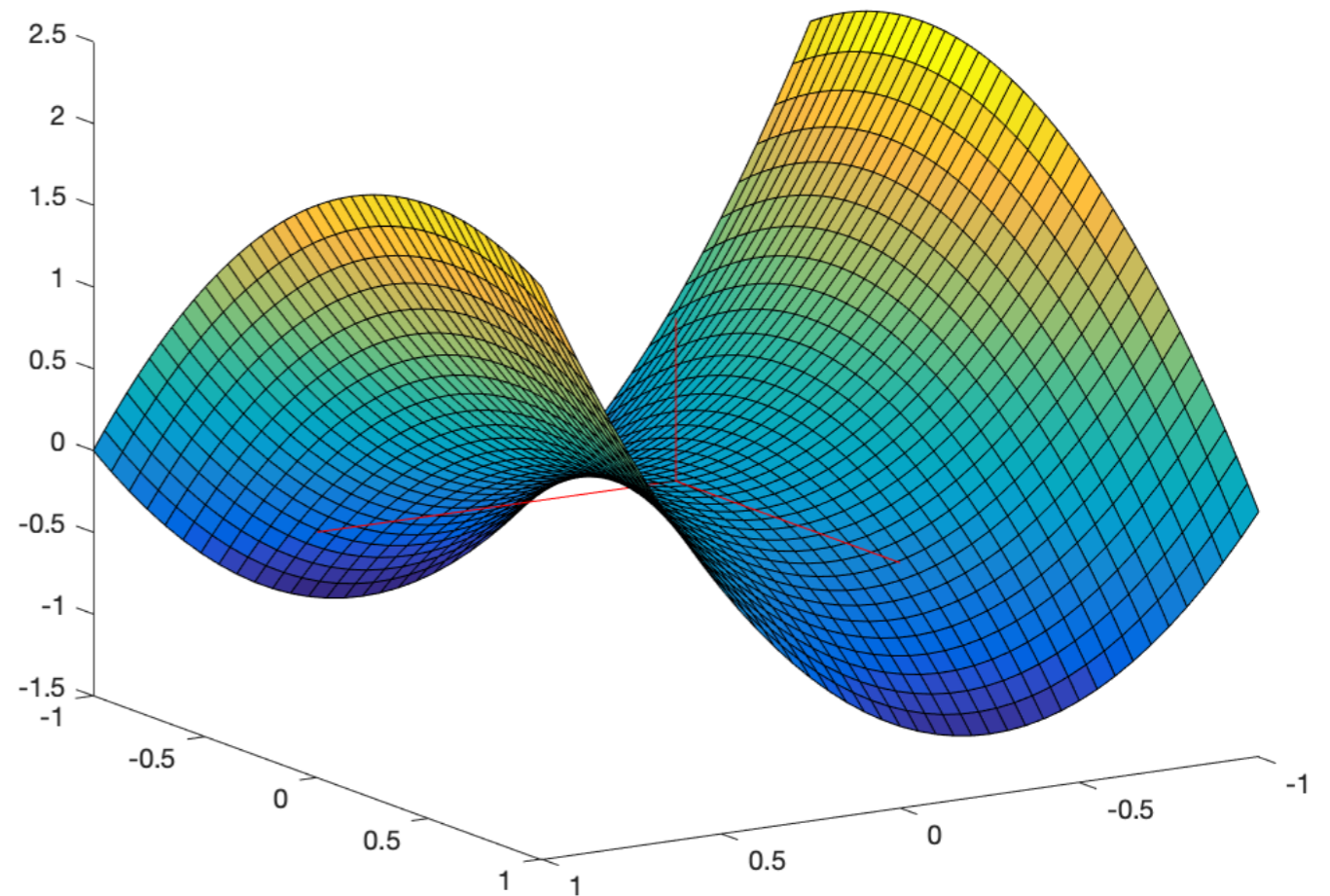
$$f: (u, v) \mapsto \begin{pmatrix} u \\ v \\ 2u^2 - v^2 + uv \end{pmatrix}$$

$$\text{Soit } P = f(0, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ 2u+v \end{pmatrix} \rightsquigarrow \frac{\partial f}{\partial u}(0, 0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial f}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ -v+u \end{pmatrix} \rightsquigarrow \frac{\partial f}{\partial v}(0, 0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{Puis } \vec{m}_P = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$





5.56	+740.21	-	0
3.24	+122.56	-	0
0.62	+140.04	-	0
.36	+180.98	-	0
.56	+740.21	-	0
.24	+122.56	-	0
.62	+140.04	-	0
.36	+180.98	-	0
.56	+740.21	-	0
.24	+122.56	-	0
.62	+140.04	-	0
.36	+180.98	-	0
.56	+740.21	-	0
.24	+122.56	-	0

# COURBURE

# « MODÉLISATION GÉOMÉTRIQUE » - GÉOMÉTRIE DES SURFACES

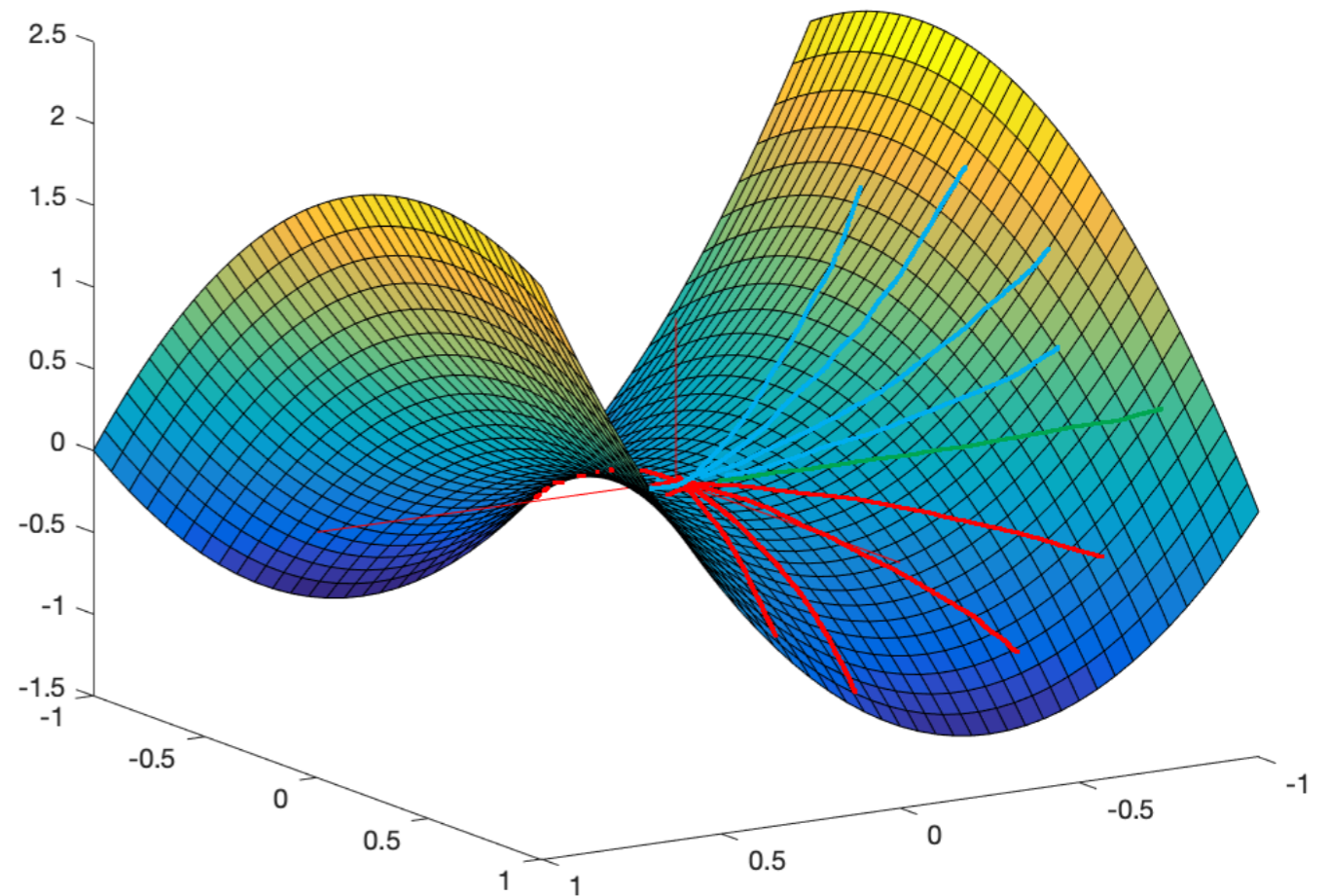
$$f: (u, v) \mapsto \begin{pmatrix} u \\ v \\ 2u^2 - v^2 + uv \end{pmatrix}$$

$$\text{Soit } P = f(0,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ 2u+v \end{pmatrix} \rightsquigarrow \frac{\partial f}{\partial u}(0,0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

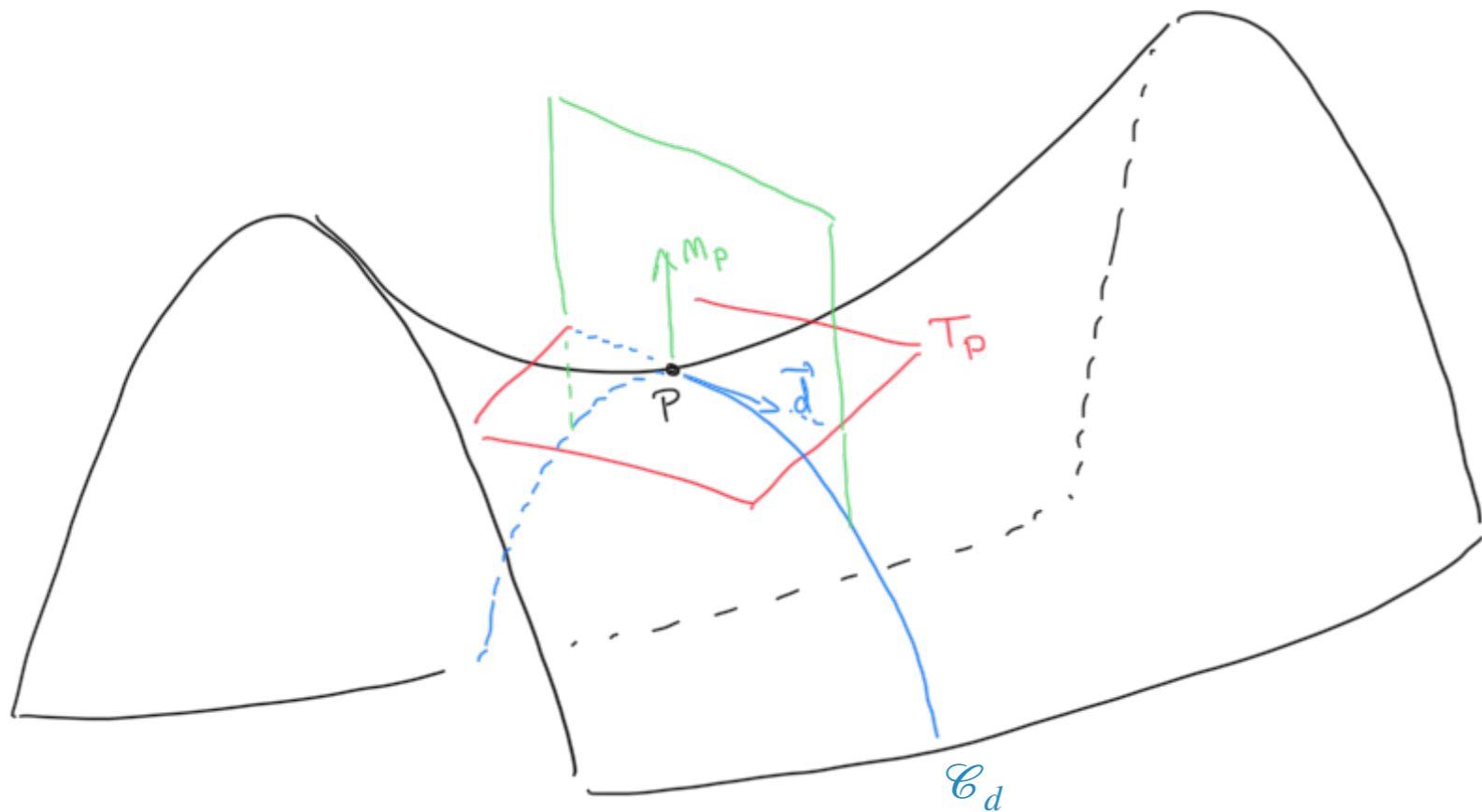
$$\frac{\partial f}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ -v+u \end{pmatrix} \rightsquigarrow \frac{\partial f}{\partial v}(0,0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{Puis } \vec{m}_P = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



Pas une mais des courbures  
(selon direction)

# COURBURE NORMALE



Courbure normale dans la direction  $\vec{d}$ :

$$k_{\vec{d}}$$

$$k_{\vec{d}}$$



Dérivées secondes de  $f$

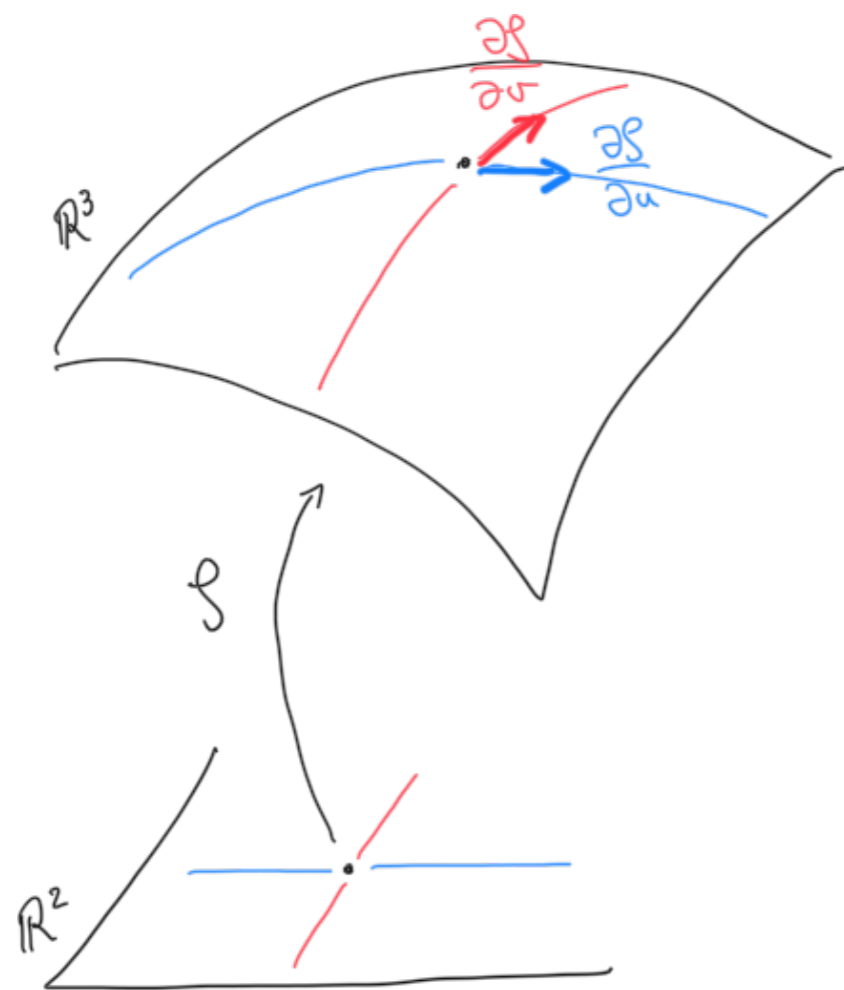
renormalisée par

Dérivées premières de  $f$

Deuxième forme fondamentale

Première forme fondamentale

# PREMIÈRE FORME FONDAMENTALE



Première forme fondamentale au point  $P$  :

$$I_P = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

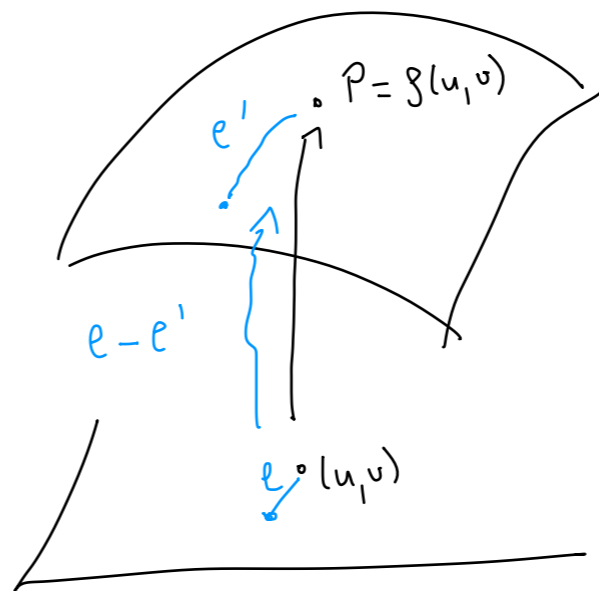
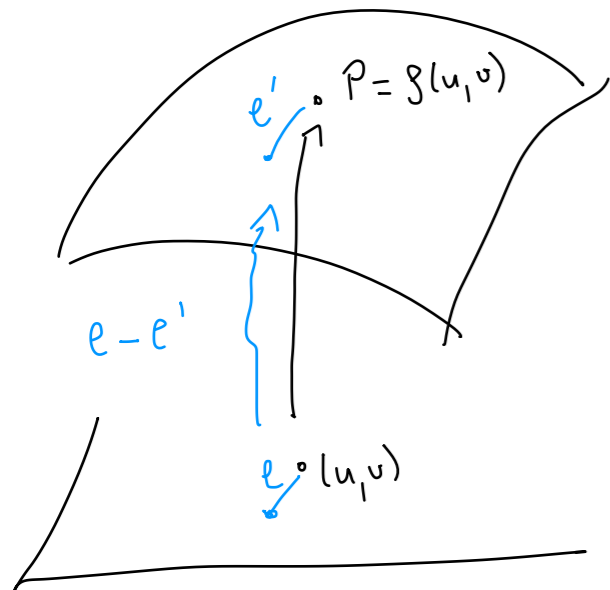
$$E = \left\langle \frac{\partial \mathcal{S}}{\partial u}, \frac{\partial \mathcal{S}}{\partial u} \right\rangle = \left\| \frac{\partial \mathcal{S}}{\partial u} \right\|^2$$

$$F = \left\langle \frac{\partial \mathcal{S}}{\partial u}, \frac{\partial \mathcal{S}}{\partial v} \right\rangle$$

$$G = \left\langle \frac{\partial \mathcal{S}}{\partial v}, \frac{\partial \mathcal{S}}{\partial v} \right\rangle = \left\| \frac{\partial \mathcal{S}}{\partial v} \right\|^2$$

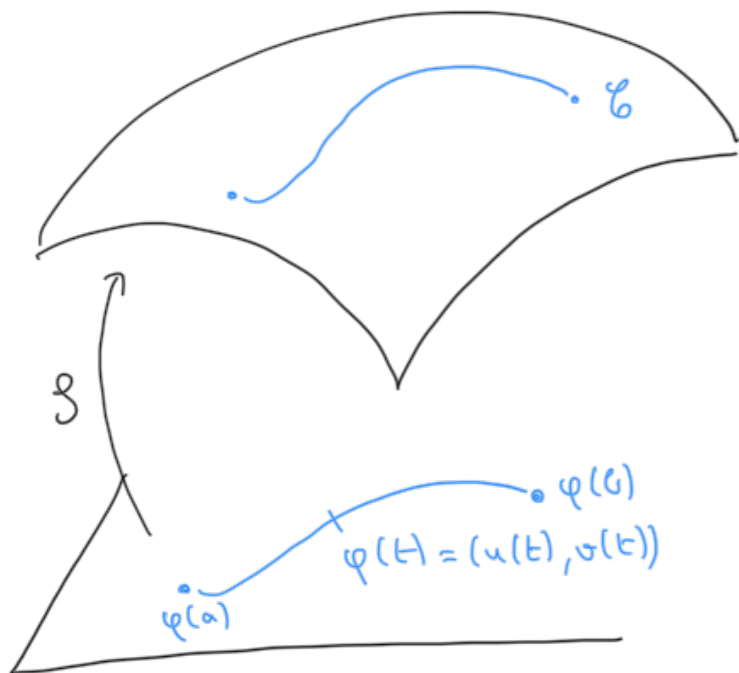


Tenseur métrique de la surface



# PREMIÈRE FORME FONDAMENTALE

Surfaces



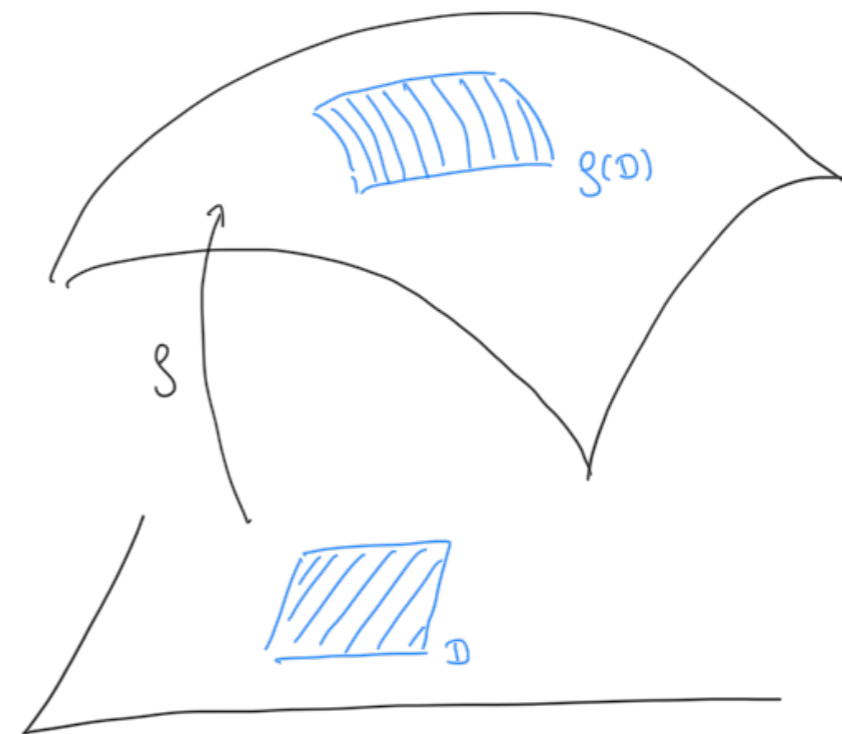
$f$  « déforme »  
les longueurs



$I_P$  mesure cette  
déformation

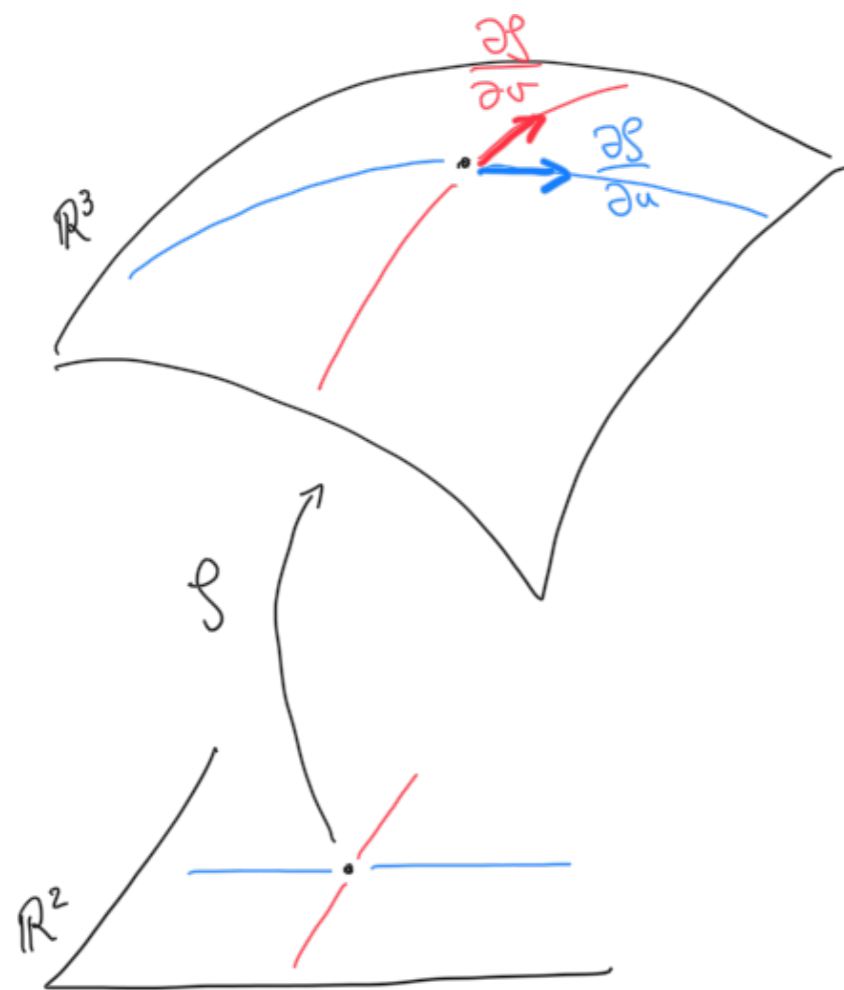
$$l(\mathcal{C}) = \int_a^b \sqrt{((u'(t), v'(t)) * I_P * (u'(t), v'(t))^t)} dt$$

Aires



$$\text{Aire}(f(D)) = \iint_D \sqrt{\det(I_P)} du dv$$

# DEUXIÈME FORME FONDAMENTALE



Deuxième forme fondamentale au point  $P$  :

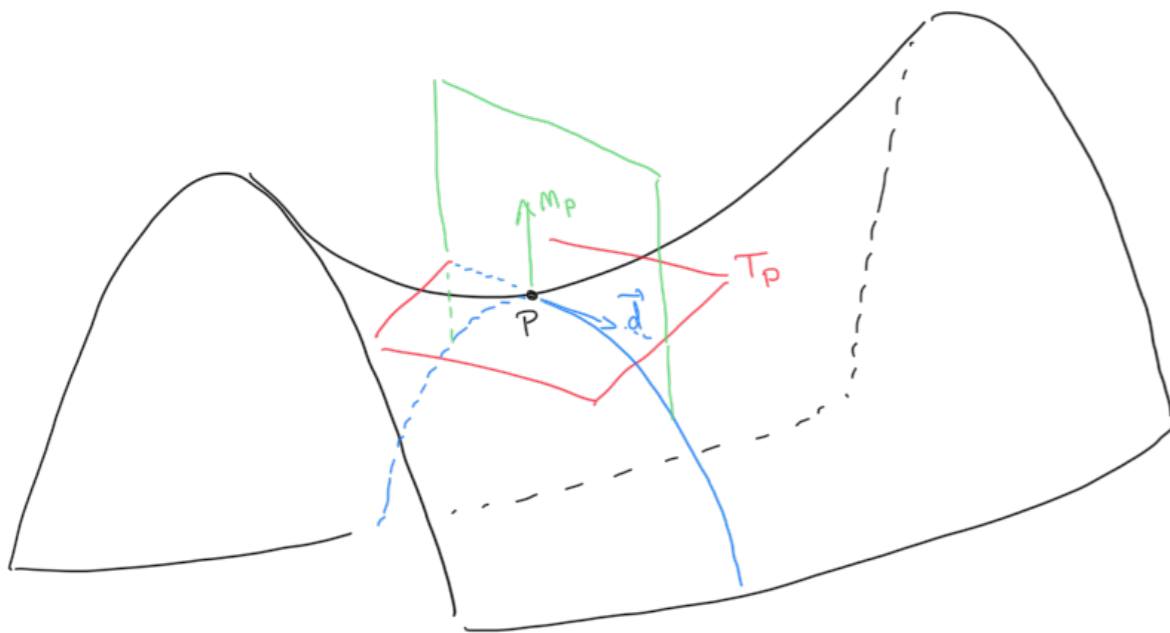
$$II_P = \begin{pmatrix} l & m \\ m & n \end{pmatrix}$$

$$l = \left\langle \frac{\partial^2 f}{\partial u^2}, \vec{n}_P \right\rangle$$

$$m = \left\langle \frac{\partial^2 f}{\partial u \partial v}, \vec{n}_P \right\rangle$$

$$n = \left\langle \frac{\partial^2 f}{\partial v^2}, \vec{n}_P \right\rangle$$

# COURBURE



La courbure normale varie comme un polynôme de degré 2 en fonction de  $d$  :

$$k_{\vec{d}} = \vec{d}^t \cdot \kappa_P \cdot \vec{d}$$



Tenseur de courbure au point  $P$

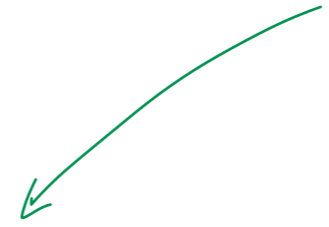
$$\kappa_P = I_P^{-1} \times II_P$$

Dans le repère  $(P, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v})$  du plan tangent

$\vec{d}$  exprimée ds la base du plan tangent :

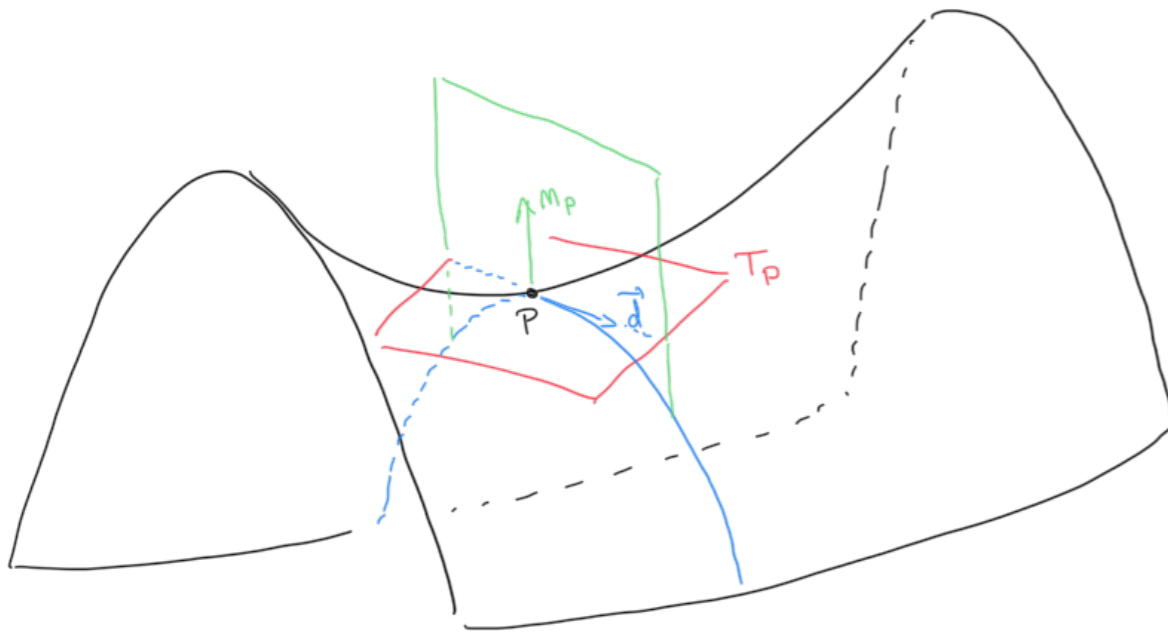
$$\begin{pmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial f}{\partial v} \end{pmatrix}$$

courbure ds direction  $\vec{d}$   
 $\leftrightarrow$   
 poly  $d = z / \vec{d}$





# COURBURE(S)



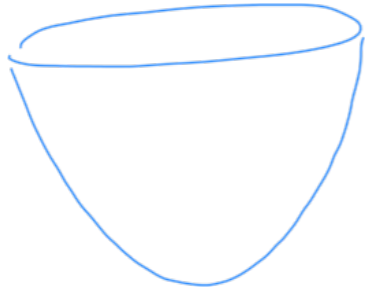

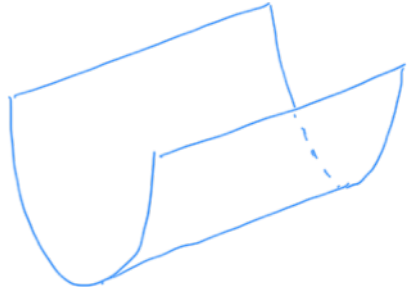

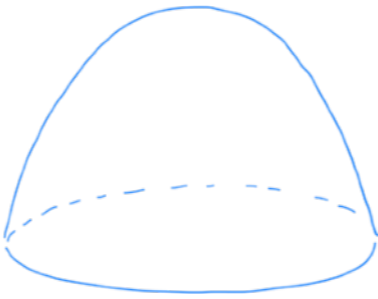
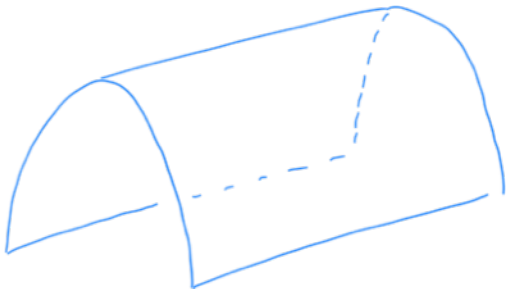
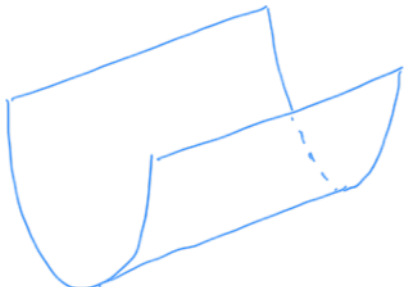
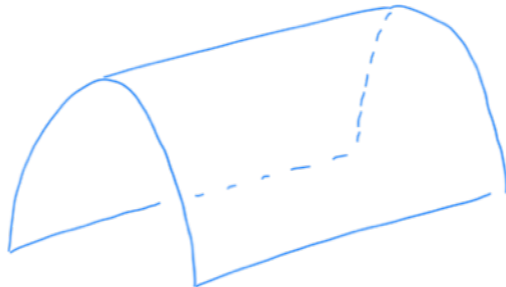

$$\kappa_P = I_P^{-1} \times II_P$$

- ▶ Matrice 2x2 symétrique
- ▶ Donc diagonalisable en base orthonormale

- ▶ Vecteurs propres :  $\vec{d}_1, \vec{d}_2$  - **directions principales**
- ▶ Valeurs propres associées :  $\vec{\kappa}_1, \vec{\kappa}_2$  - **courbures principales**
  - $\kappa_i > 0$  : convexe selon  $\vec{d}_i$
  - $\kappa_i = 0$  : plat selon  $\vec{d}_i$
  - $\kappa_i < 0$  : concave selon  $\vec{d}_i$

Extréma des courbures

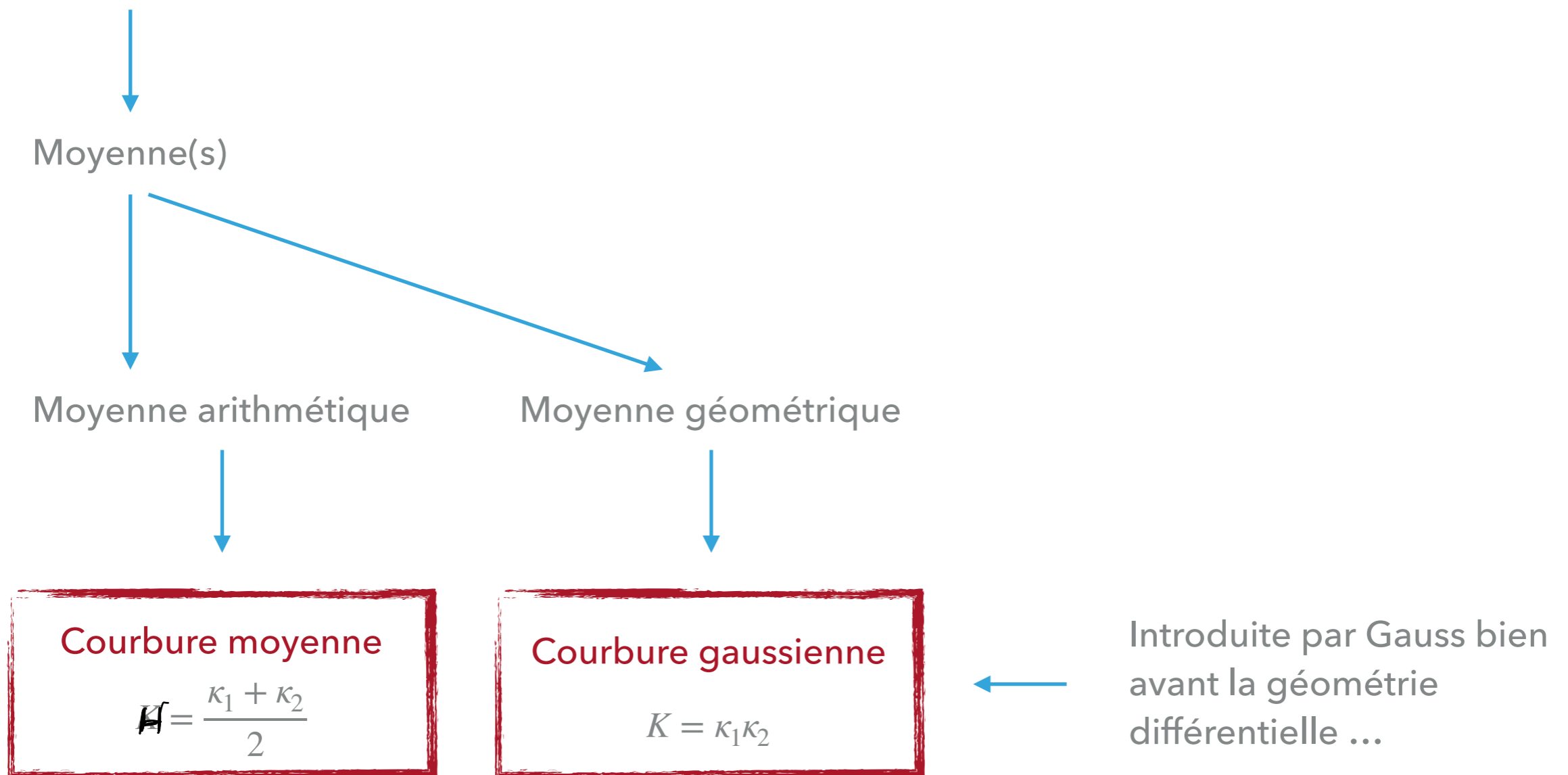
# COURBURE(S)

$\begin{matrix} \vec{\kappa}_1 \\ \vec{\kappa}_2 \end{matrix}$	$> 0$	$< 0$	$= 0$
$> 0$			
$< 0$			
$= 0$			

Classification locale

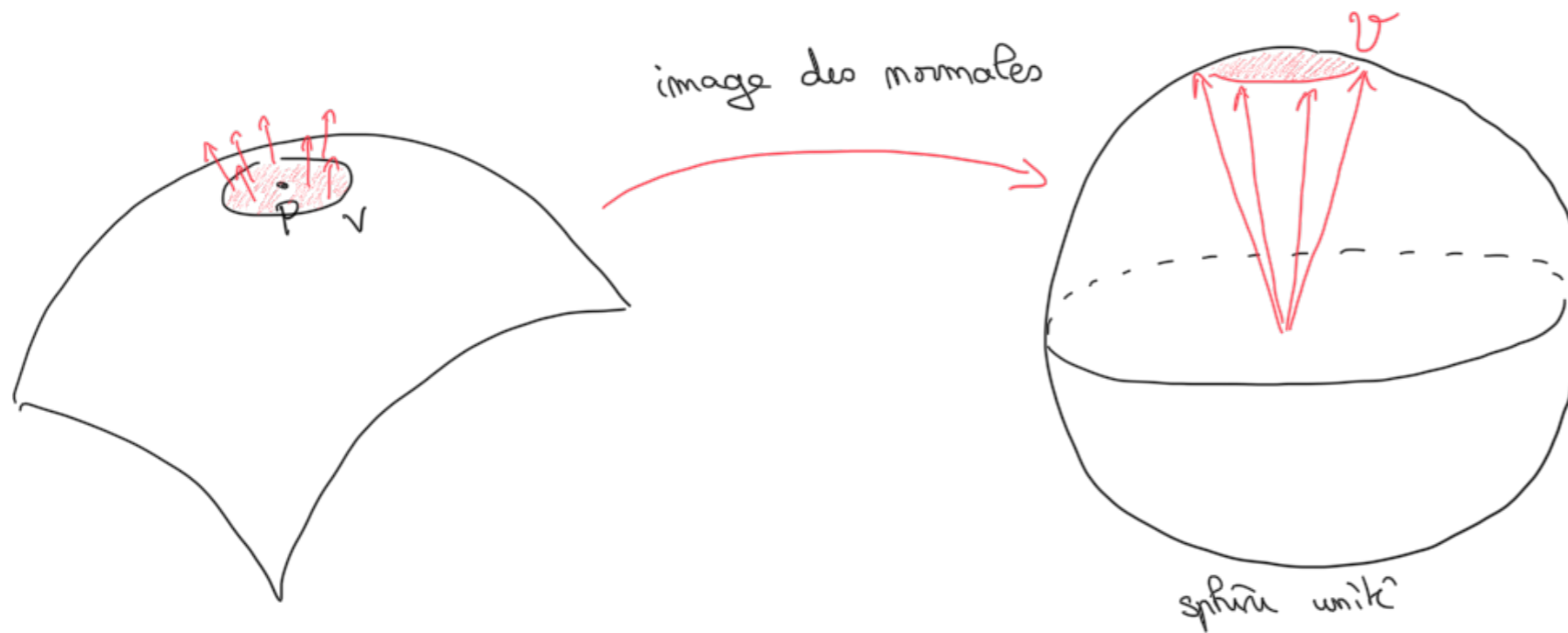
# COURBURE(S)

Deux mesures ... difficile à manipuler (ex : carte de courbure, critère algorithmique de courbure ...)



# COURBURE GAUSSIENNE

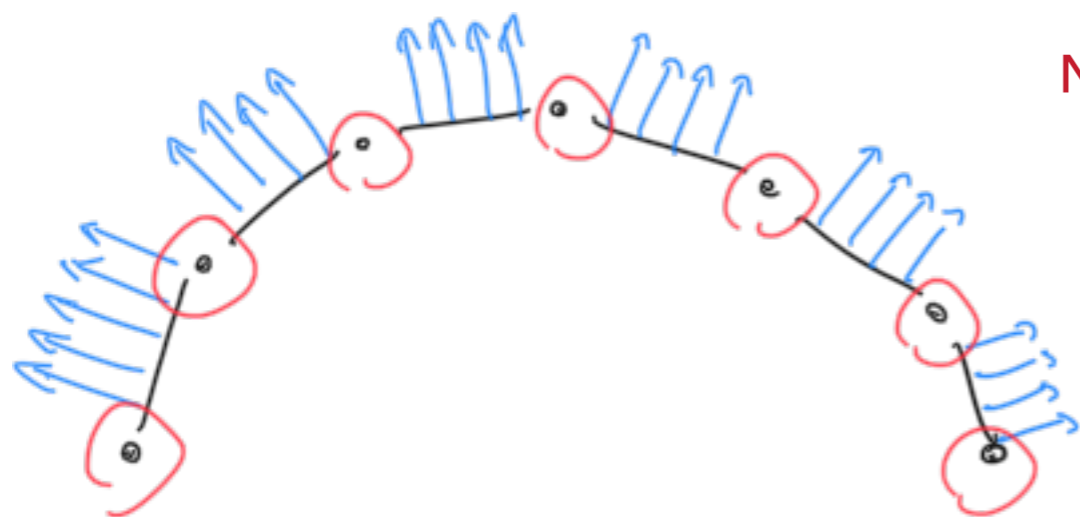
Theorema Egregium de Gauss



$$\lim_{V \rightarrow \{P\}} \frac{\text{Aire}(\mathcal{V})}{\text{Aire}(V)} = \kappa_P$$

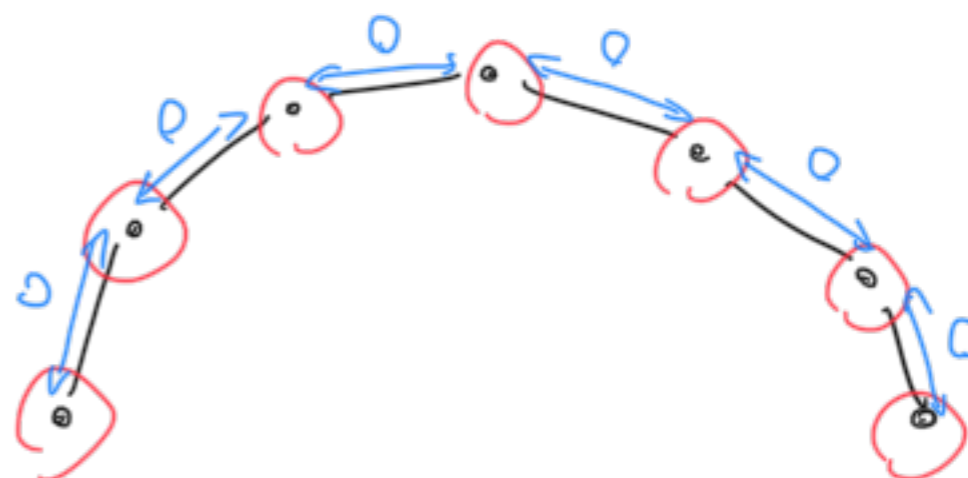
# GÉOMÉTRIE DES MAILLAGES

## NORMALES/COURBURES D'UN MAILLAGE ?



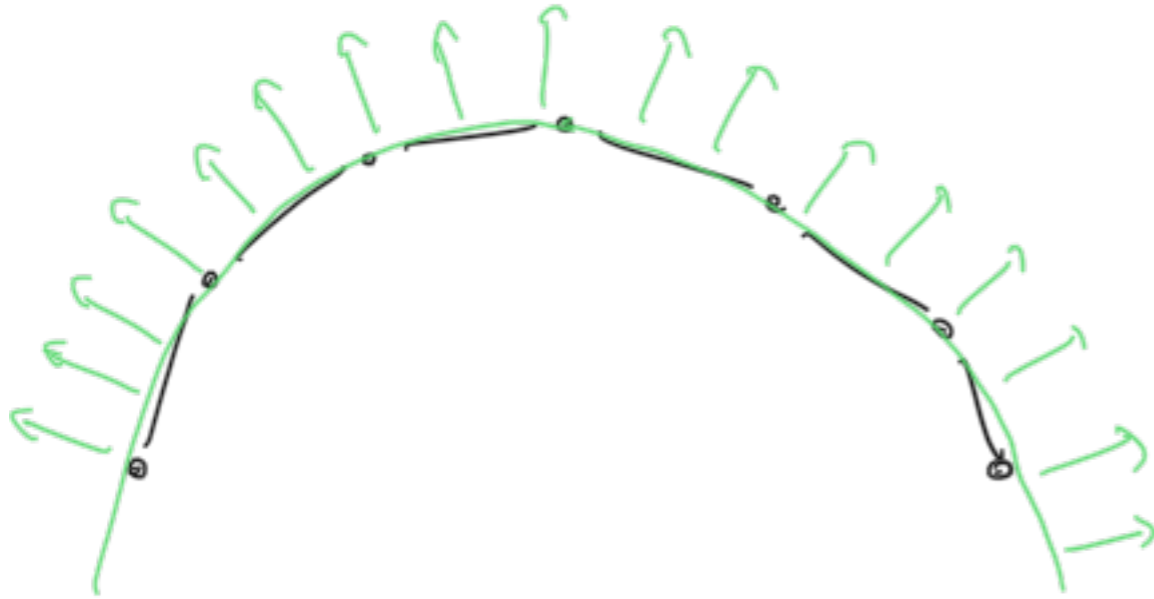
Normales

Non défini



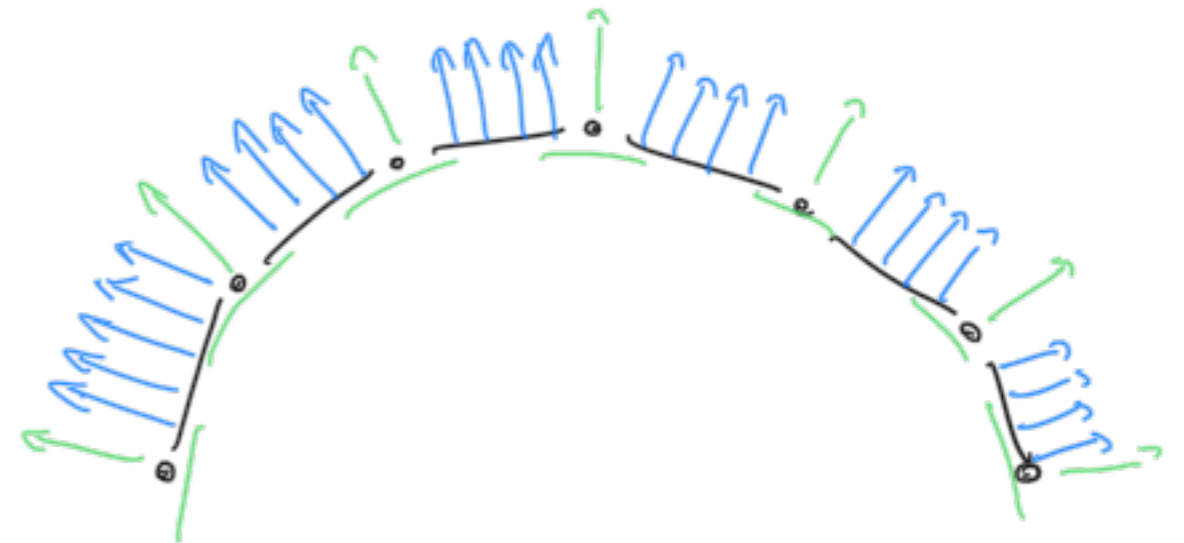
Courbure(s)

## NORMALES/COURBURES D'UN MAILLAGE ?



Intuition ...

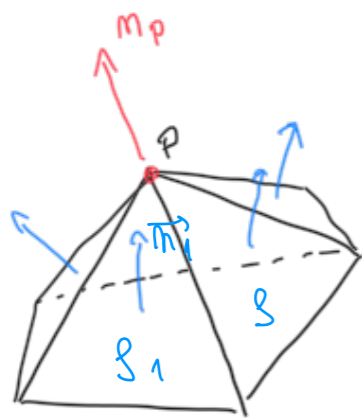
Le maillage est la  
discrétisation d'une surface  
continue



**Pb : calculer des valeurs approchées cohérentes**

→ approximations / heuristiques

# ESTIMATION DES NORMALES



$S_i \rightsquigarrow \vec{m}_i$  normale

On estime généralement la **normale au point  $P$**  comme :

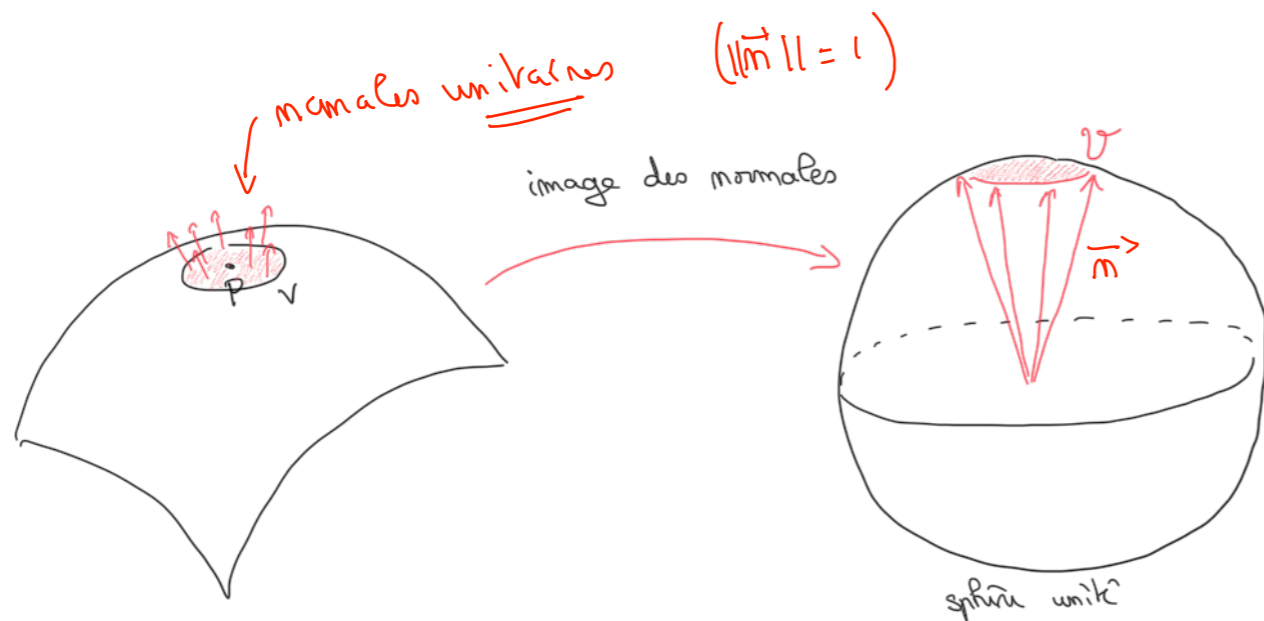
- ▶ Moyenne des normales des faces voisines
- ▶ Eventuellement pondérée par l'aire des faces

$$\vec{m}_j = \frac{\sum_{i=1}^N \vec{m}_i}{N}$$

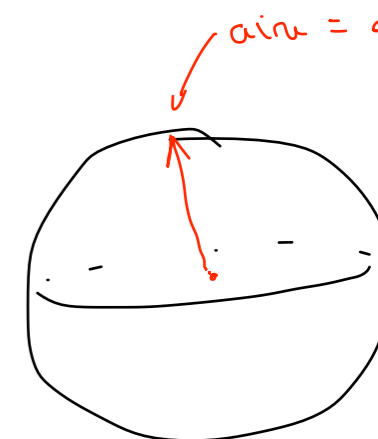
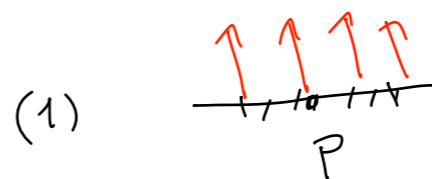
$$\frac{\sum_{i=1}^N \text{aire}(S_i) \times \vec{m}_i}{\sum \text{aire}(S_i)}$$



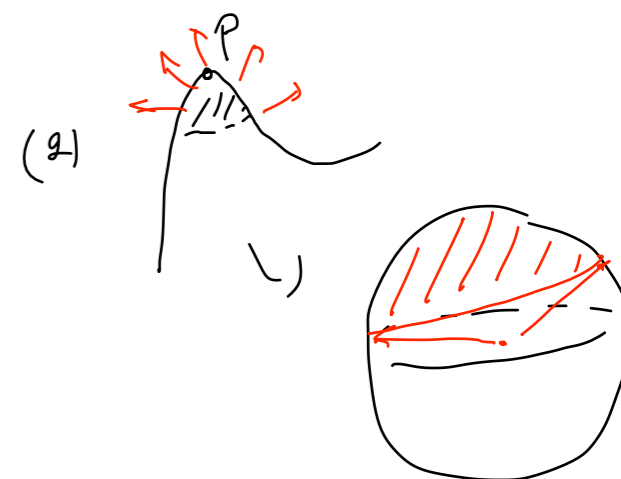
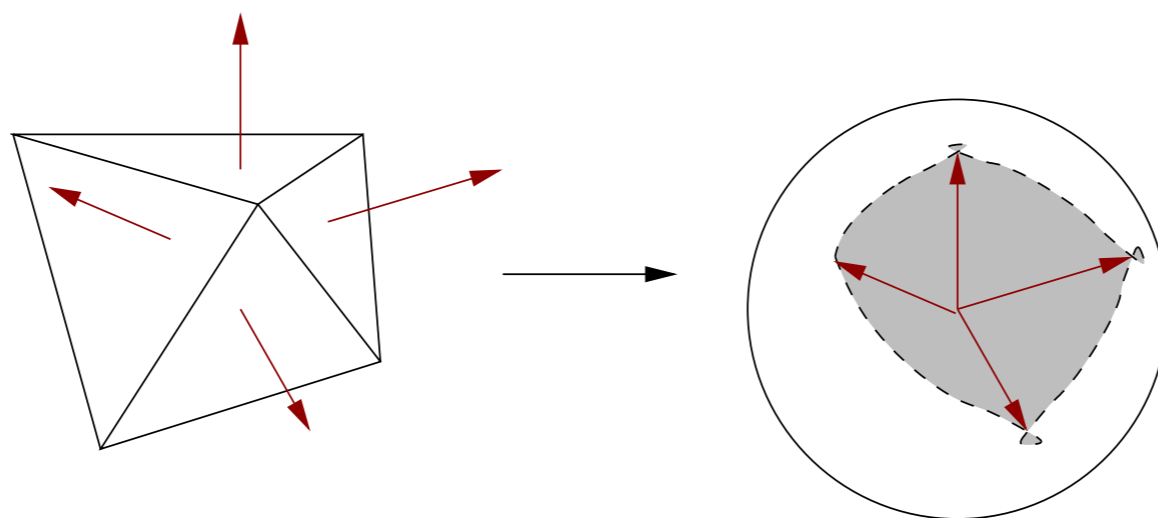
# COURBURES : TH. EGREGIUM DE GAUSS ET DÉFAUT ANGULAIRE



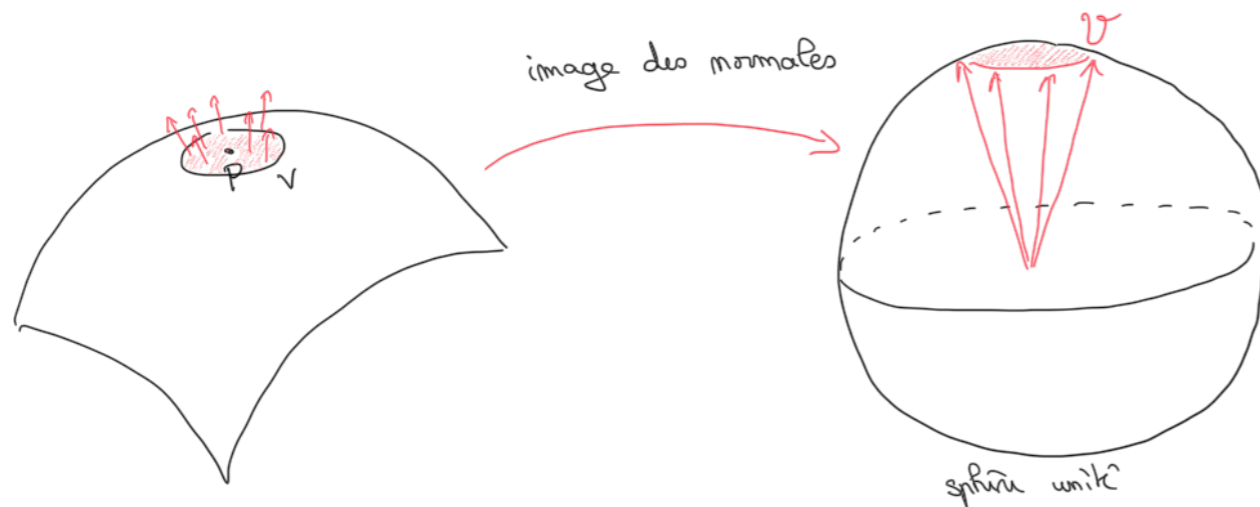
$$\lim_{V \rightarrow \{P\}} \frac{\text{Aire}(V')}{\text{Aire}(V)} = \kappa_P$$



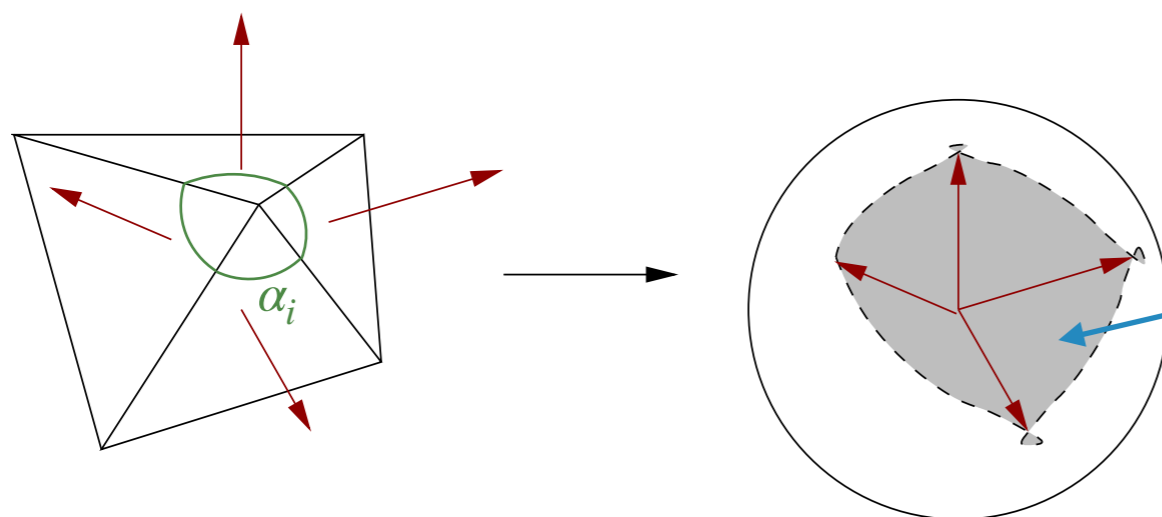
$\hookrightarrow \kappa_P = 0 = \kappa_1 \times \kappa_2$



# COURBURES : TH. EGREGIUM DE GAUSS ET DÉFAUT ANGULAIRE



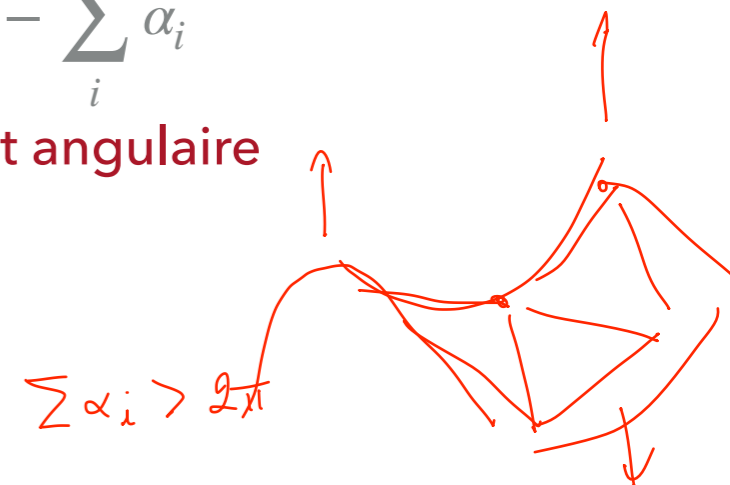
$$\lim_{V \rightarrow \{p\}} \frac{\text{Aire}(\mathcal{V})}{\text{Aire}(V)} = \kappa_p$$



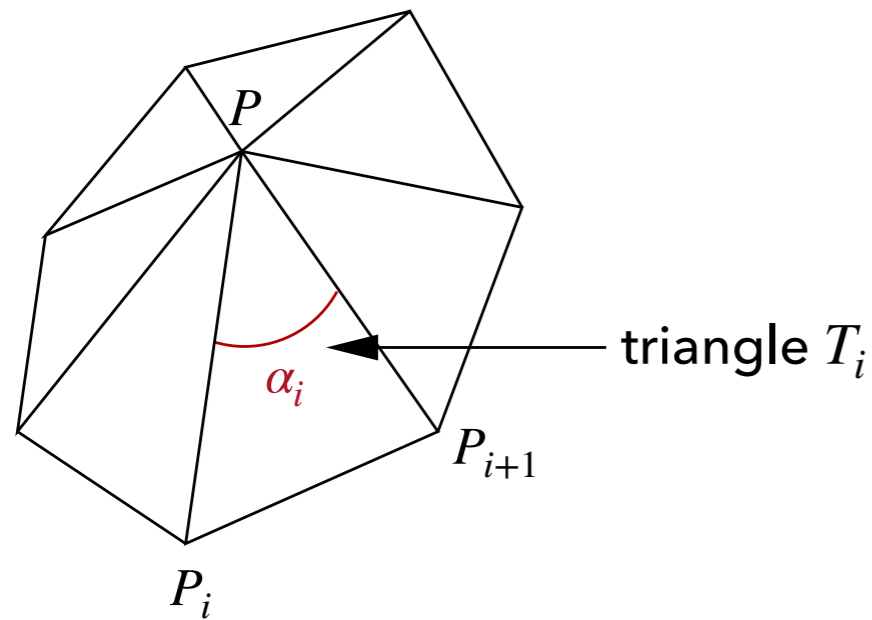
Valeur de cette aire sphérique :

$$2\pi - \sum_i \alpha_i$$

Défaut angulaire



## COURBURE GAUSSIENNE « CLASSIQUE » (REGGE)



Approximation de la courbure Gaussienne :

$$K_P = \frac{2\pi - \sum_i \alpha_i}{\sum_i \text{Aire}(T_i)} \quad \text{au point } P$$

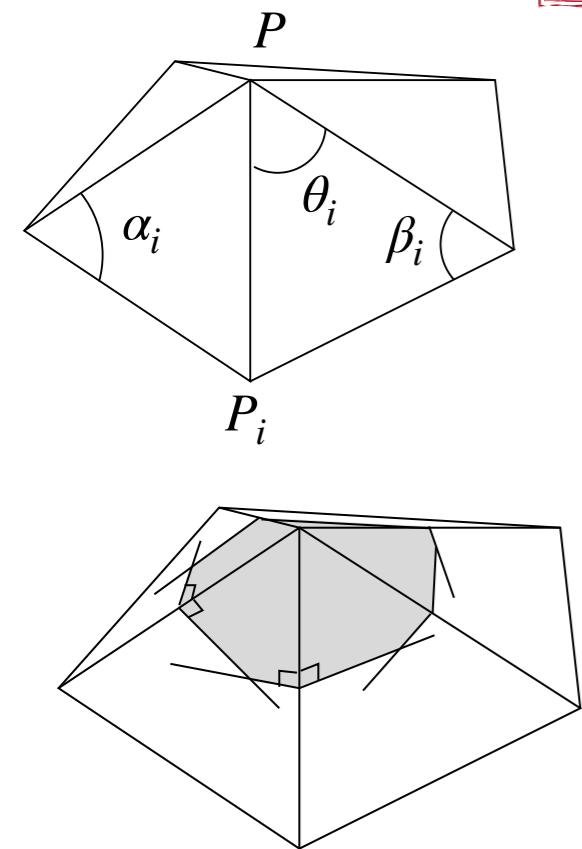
- ▶ C'est la formule que vous avez implémentée au TP1
- ▶ Peu précise, dépend du maillage
- ▶ Ne converge pas

Il existe aussi des approximations basiques et géométriques de la courbure moyenne ...

## APPROXIMATIONS PLUS FINES

On the angular defect of triangulations and the pointwise approximation of curvatures  
V. Borrelli, F. Cazals, J.-M. Morvan

Discrete Differential-Geometry Operators for Triangulated 2-Manifolds  
M. Meyer, M. Desbrun, P. Schröder, and A. Barr



$$H_P \cdot N_P = \frac{1}{2 \text{Aire}_{\text{mixte}}} \sum_i (\cot \alpha_i + \cot \beta_i) (P - P_i)$$

$$K(P) = \frac{2\pi - \sum_i \theta_i}{\text{Aire}_{\text{mixte}}}$$

La convergences n'est pas garantie :

- dépend de la triangulation  
plus elle est régulière, meilleure est la convergence
  - idéal : régulière, valence 6
- sur une triangulation irrégulière, non garantie
  - au mieux, convergence en norme  $\mathcal{L}_1$

## AUTRES APPROCHES

On the angular defect of triangulations and the pointwise approximation of curvatures  
V. Borrelli, F. Cazals, J.-M. Morvan

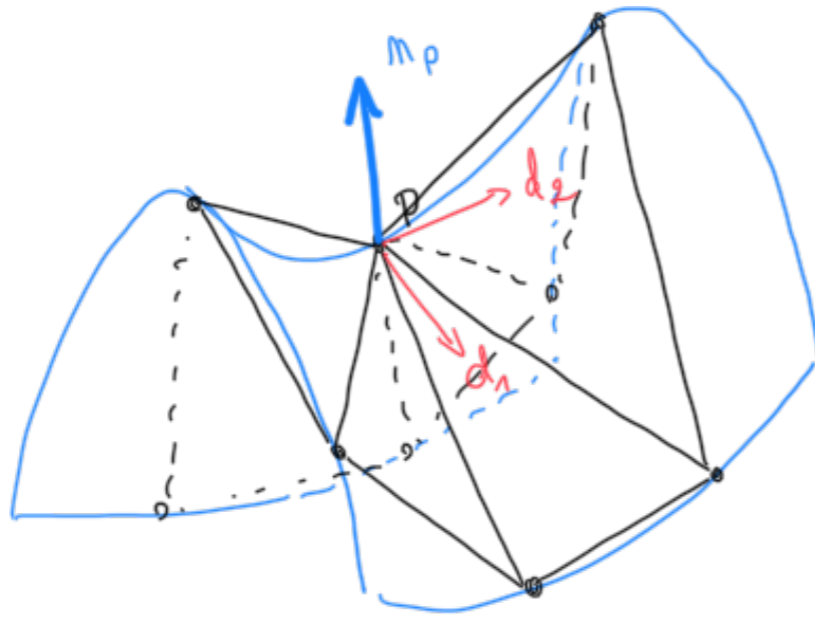
Discrete Differential-Geometry Operators for Triangulated 2-Manifolds  
M. Meyer, M. Desbrun, P. Schröder, and A. Barr

Estimating the Tensor of Curvature of a Surface from a Polyhedral Approximation  
G. Taubin



Estimation de  $K_p$  (tenseur de courbure)

## CALCUL VIA UNE SURFACE CONTINUE



Ajuster une surface lisse  
(paramétrique ou cartésienne)  
au voisinage de  $P$



Calculer normale et courbures  
de la surface lisse

- ▶ Méthode **plus lente** mais **plus stable**
- ▶ La question est alors de bien approcher localement le maillage