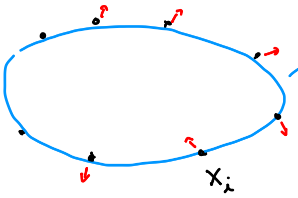
TD / TP 3

Ex. 3



g : $a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz + a_7x + a_8y + a_9z = 1$

inconnues : a_i ⟷ surface

↓

9 inconnues.

syst — 9 inconnues

↓

9 éq ⟷ $\boxed{9 \text{ pts}}$ X_i

Syst :

$$\forall X_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \rightarrow \left\{ \begin{array}{l} \vdots \\ a_1x_i^2 + a_2y_i^2 + a_3z_i^2 + a_4x_iy_i + a_5x_iz_i + a_6y_iz_i + a_7x_i + a_8y_i + a_9z_i = 1 \quad \leftarrow i \\ \vdots \end{array} \right.$$

9 éq — 9 inconnues

$$w = \begin{pmatrix} a_1 \\ \vdots \\ a_9 \end{pmatrix}$$

syst. linéaire

↓

$$Xw = y$$

$$Xw = y \qquad X = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_i^2 & y_i^2 & z_i^2 & x_iy_i & x_iz_i & y_iz_i & x_i & y_i & z_i \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \text{ — } 9 \times 9$$

$$y = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \text{ — } 9 \times 1$$

Résoudre en Matlab : $w = X \backslash y$

Pts ⇝ matrice M par col

9 pts → M : $\begin{array}{|c|c|c|c|} \hline \; & \; & X_i & \; \\ \hline \end{array}$ ⟵ M = rand(3,9)

$1 \quad \ldots \quad i \quad \ldots \quad 9$

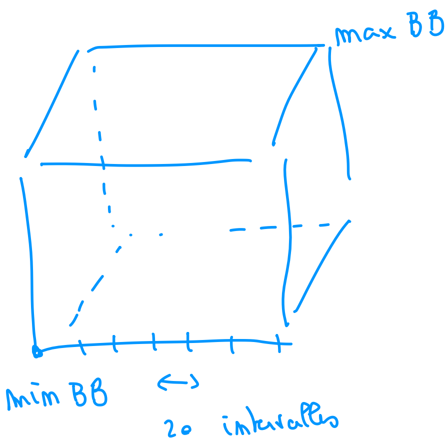
plotImplicit3D (f , minBB , maxBB , step BB)

f : prenant en arg X, Y, Z vects de coords et calculant f en // (.* ....)

minBB, maxBB : pt3D : limites de la boite englobante de représentation

step BB : pt_entier_3D : nbre de seg. selon x, y, z

plotImplicit3D(f, [-1, -1, -1], [1, 1, 1], [20, 20, 20])



Script

```
clear
clc
clf
hold on

addpath('MarchingCubes')

N = 9 ;
M = rand(3,N) ;
eps = 10/100. ;

f = compute_quad(M) ;

% Représentation graphique
window = 1.5 ;

plotImplicit3D(f, [-window, -window, -window], [window, window, window], [20, 20, 20]) ;
plot3(M(1,:), M(2,:), M(3,:), '.g', 'MarkerSize',20) ;

% Perturbation des points

delta = (rand(size(M))-.5)*2*eps ;
M = M+delta ;
f2 = compute_quad(M) ;

% Représentation graphique

plotImplicit3D(f2, [-window, -window, -window], [window, window, window], [20, 20, 20]) ;
plot3(M(1,:), M(2,:), M(3,:), '.r', 'MarkerSize',20) ;
```

Function compute_quad

```
function [f] = compute_quad(M)
    N = size(M,2) ;
    X = [(M(1,:)').^2, (M(2,:)').^2, (M(3,:)').^2, (M(1,:)').*(M(2,:)'), (M(1,:)').*(M(3,:)'), ...
    (M(2,:)').*(M(3,:)'), M(1,:)', M(2,:)', M(3,:)'] ;
    y = ones(N, 1) ;
    disp(cond(X))

    % Résolution
    w = X\y ; % a1, a2 .......

    % Reconstruction de la fonction f

    f = @(x,y,z) w(1)*x.^2 + w(2)*y.^2 + w(3)*z.^2 + w(4)*x.*y + w(5)*x.*z + …
            w(6)*y.*z + w(7)*x + w(8)*y + w(9)*z - 1;
end
```

TD → ex 1, 2, 5

GSL → GNU scientific library ⟷ Matlab C/C++

Linear algebra – $\boxed{\text{Eigen}}$ ⟶ $\boxed{\text{BLAS}}$

Matrice   C/C++/JAVA/Python ....

Ex. 1

1) We get $A = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ $d_1 = 10^{-3}$, $d_n = 10^5$

$\text{cond}(A) \; ? = \frac{10^5}{10^{-3}} = 10^8$

Gauss / LU

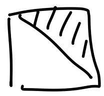
Choleski / $R.R^t$ (lower, upper)

Householder / QR

2) $[Q,R] = qr(A)$

↓

$A = Q \cdot R$ — Q: orthogonal, R: upper

$A \rightsquigarrow$ 

$\underbrace{Q_n \times \cdots \times Q_1}$ orthogonal / symmetries

$M = rand(n,n)$

$[Q,R] = qr(M)$

↳ random orthogonal matrix ...

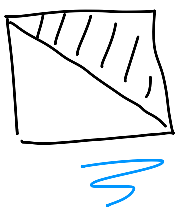
3) $B = Q \times A$

$\text{cond}(B) = \text{cond}(A)$ / 10^8 ⟵ can be chosen through the val. on the diag of A

solve syst.

B ⟶ Gauss → LU, Householder → QR

cond ⩾



cond ?

cond(B)

4) Cholesky $\longrightarrow$ ~~symmetric~~ definite positive matrices

$B = Q \times A$

ortho (Q), diag (A)

$B \overset{?}{=} B^t = A^t \times Q^t$, $A^t = A$

$\underset{\text{ortho}}{Q^t} = Q^{-1} \neq Q$

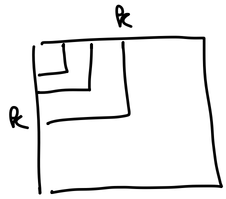
5) A — diag — $\mathrm{cond}(A) = c$ $\longrightarrow$ $A^t = A$

$B = Q \times A \times Q^t \rightarrow$ symmetric ✓

$\overset{M}{\text{Definite positive}} \xrightarrow{\text{def}} X^t \cdot M \cdot X > 0 \quad \forall X \neq 0$

$\rightarrow$ eigenvalues of $^tX \cdot M \cdot X > 0$

$\rightarrow$ det(sub $k \times k$) > 0

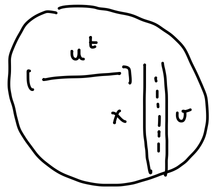


Let $X \in \mathbb{R}^n$

$(*)\ X^t \cdot (\underset{\text{orthogonal}}{Q} \cdot \overset{\text{diag } \oplus}{\boxed{A}} \cdot Q^t) \cdot X > 0$?

t

$u^t \times v = \langle u, v \rangle$



$\rightarrow u^t \times u = \langle u, u \rangle = \|u\|^2$

$$A = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} = \underbrace{\begin{pmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & \sqrt{d_n} \end{pmatrix}^t}_{R} \times \begin{pmatrix} \sqrt{d_1} & & \\ & \ddots & \\ & & d_n \end{pmatrix}$$

$d_i > 0$

$$(*) = \underbrace{\boxed{X^t \cdot Q \cdot R^t}}_{u^t} \underbrace{\boxed{R \cdot Q^t \cdot X}}_{u} = u^t \times u = \|u\|^2 \geq 0$$

$= \, ?$

$u = 0$

$$\cancel{R} \cdot \cancel{Q^t} \cdot X = 0$$

$$\underbrace{R}_{\text{diag}} = \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} \Downarrow R^{-1} \checkmark$$

Q is orthogonal

$\Downarrow$

$Q^t = Q^{-1}$

invertible

$$u = 0 \iff X = 0$$

The matrix we have built is definite positive.

$$B = Q \times A \times Q^t \longrightarrow \text{random sym. def positive matrix cond. } c$$

Ex. 2

$A \rightsquigarrow$ eigenvalues $|\lambda_1| > |\lambda_2| \; \dots \; > |\lambda_d|$

A diagonalizable

$u_1 \; \dots \; u_d$ basis

$$x_0 = \sum_i \alpha_i \cdot \vec{u}_i \qquad \lambda_i$$

$$x_n \, ? \quad = A \cdot x_{n-1}$$

$$\begin{aligned} x_1 &= A \cdot x_0 \\ &= A\Big(\sum_i \overset{\mathbb{R}}{\alpha_i} \, u_i\Big) \\ &= \sum_i \alpha_i \cdot \underbrace{A\vec{u}_i}_{= \lambda_i \cdot \vec{u}_i} \\ &= \sum_i \alpha_i \cdot \lambda_i \cdot u_i \end{aligned}$$

$$x_2 = A x_1 = A\left(\sum_i \overset{\mathbb{R}}{\alpha_i}\, \overset{\mathbb{R}}{\lambda_i}\, u_i\right) = \sum_i \alpha_i \cdot \lambda_i \cdot \underbrace{A u_i}_{\lambda_i u_i} = \sum_i \alpha_i \cdot \lambda_i^2\, u_i$$

$\vdots$

$$x_n = \sum_i \alpha_i \cdot \lambda_i^n\, u_i \qquad \lambda_1 > \lambda_2 \cdots\cdots > \lambda_d$$

Then $$\frac{x_n}{\lambda_1^n} = \sum_i \alpha_i \left(\frac{\lambda_i}{\lambda_1}\right)^n \cdot u_i \qquad \left(\frac{\lambda_i}{\lambda_1}\right)^n \xrightarrow[n\to\infty]{} \begin{cases} 1 & i=1 \\ 0 & i>1 \end{cases}$$

$\lambda_i < \lambda_1$

$$\frac{x_n}{\lambda_1^n} \xrightarrow[n\to\infty]{} \alpha_1 \underbrace{u_1}_{\text{first eigen vector.}}$$

Random diagonalizable matrix

$\to$ rand diag matrix A

$\to$ rand base change

$\updownarrow$

invert. matrix P

change basis $\leadsto$ $P^{-1} \times A \times P$