§1. Géométrie 3D

(surfaces - courbes)

→ calcul différentiel
géométrie différentielle

Surface

→ Maillages — Discrets

- - - - - - - - - - - - - - - - - -

→ Surfaces implicites (*)
$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ ~ surf. $\{x \text{ tq } f(x) = \overset{c}{0}\}$

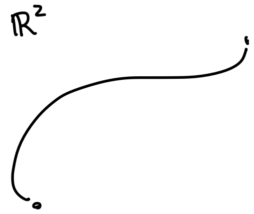
Continus → Surfaces paramétriques
$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

(**) $(u,v) \longmapsto f(u,v) - \begin{vmatrix} x_{u,v} \\ y_{u,v} \\ z_{u,v} \end{vmatrix}$

Courbes

$f(t)$ — Paramétriques

$f : \underset{\text{param } t}{\mathbb{R}} \longrightarrow \mathbb{R}^2$



(*) 

$\boxed{x^2 + y^2 + z^2 - 1} = 0$
$f(x,y,z)$

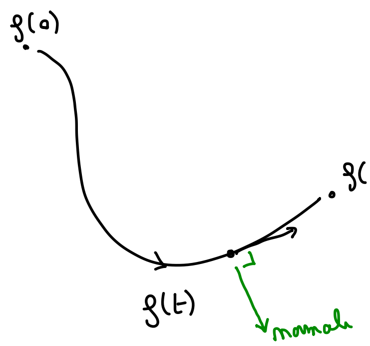
(**) Tore

$(u,v) \longmapsto \begin{cases} (r + R\cos v)\cos u \\ (r + R\sin v)\cos u \\ r.\sin v \end{cases}$

Modèles continus
→ caractérisation de la géom.
par des prop. différentielle

~> modèles paramétriques ⟹ formules.

I. Intuitions sur les courbes (param).
|
lisse



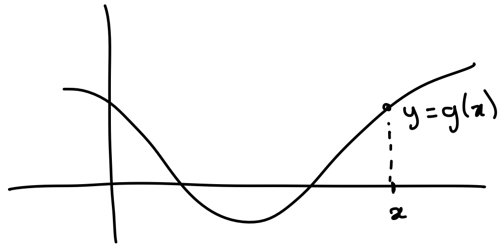
équation paramétrique

$f : [0,1] \longrightarrow \mathbb{R}^2 = \begin{vmatrix} f_1(t) & - x \\ f_2(t) & - y \end{vmatrix}$
t

① → normale / tangente ~> liée à $f'(t)$
$\quad\quad |$
$\quad\quad f'$

$\quad = \begin{pmatrix} f'_1(t) \\ f'_2(t) \end{pmatrix}$ vecteur tangent

$y = g(x)$



Équations cartésiennes

éq. cart.

→ tangente : $g'(x)$ — pente de la tangente

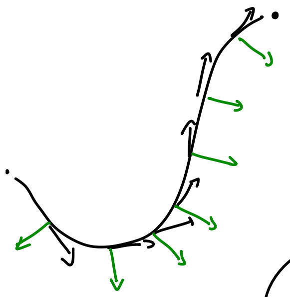
→ éq. paramétrique

$$\mathbb{R} \longrightarrow \mathbb{R}^2$$
$$x \longmapsto \begin{pmatrix} x \\ g(x) \end{pmatrix}$$

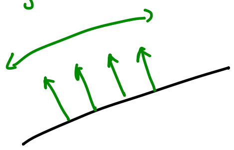
→ éq. implicite

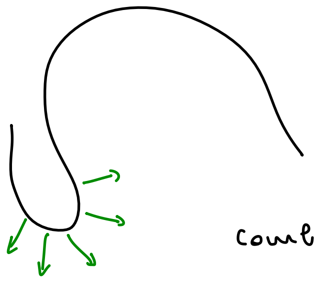
$$\underbrace{g(x) - y}_{f(x,y) = g(x) - y} = 0$$

② → courbure



faible variation $\vec{n}$





courbure

forte variation $\vec{n}$

$\Downarrow$

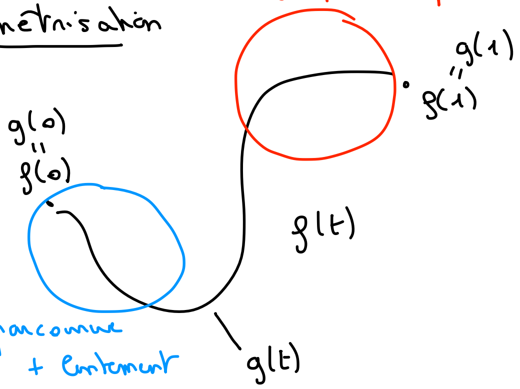
dérivée importante de $\vec{n} \sim f'$

courbure $\sim f''$

paramétrisation

Paramétrisation

g plus rapide que f



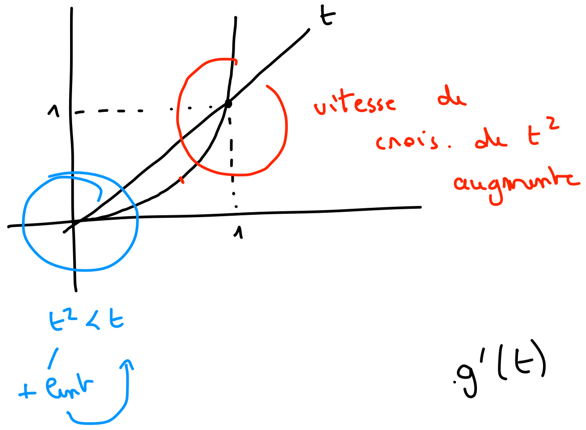
$$g : [0,1] \longrightarrow \mathbb{R}^2$$
$$t \longmapsto f(t^2)$$

$g(0) = f(0)$

$g(1) = f(1)$

$g(t) = f(t^2)$

$g(0{,}1) = f(0{,}01)$ ← même pt atteint à des moments ≠



$\left\{\begin{array}{l}\text{tangente} \\ \text{normale}\end{array}\right.$ — dérivée

courbure — " seconde

$$g'(t) = \left(f(t^2)\right)' = 2t \cdot f'(t^2)$$

$$g''(t) = 2 \cdot f'(t^2) + 4t^2 \cdot f''(t^2)$$

tangente modulo paramétrisation $\cancel{f'(t)}$ $f'(s)$

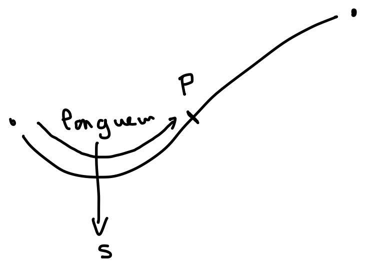
courbure " $\cancel{f''(t)}$ $f''(s)$

parmi toutes les paramétrisations
une est unique
↓
parcours à vitesse constante

⇝ paramétrisation normale
⇝ abscisse curviligne (s)
$\cancel{t}$ s

g une autre vitesse de parcours

$$f'(t) = \frac{g'(t)}{|g'(t)|}$$



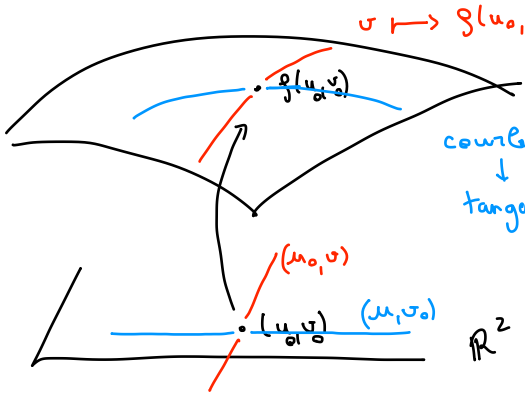
II. Surfaces (paramétriques)

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(u,v) \longmapsto f(u,v) \in \mathbb{R}^3$$

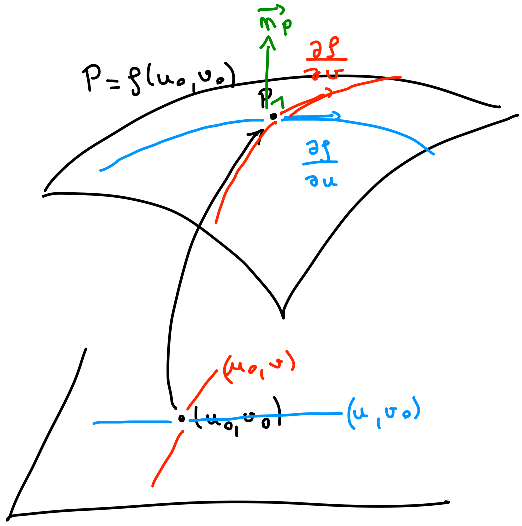
① Plan tangent / normale

tangente $(v \mapsto f(u_0,v))' = \dfrac{\partial f}{\partial v}$

$v \mapsto f(u_0,v)$



courbe: $u \mapsto f(u,v_0)$
↓
tangente $(u \mapsto f(u,v_0))' = \dfrac{\partial f}{\partial u}$



Les tangentes au pt. P forment un plan

↓

engendré par $\frac{\partial \rho}{\partial u}, \frac{\partial \rho}{\partial v}$

Normale au pt P:

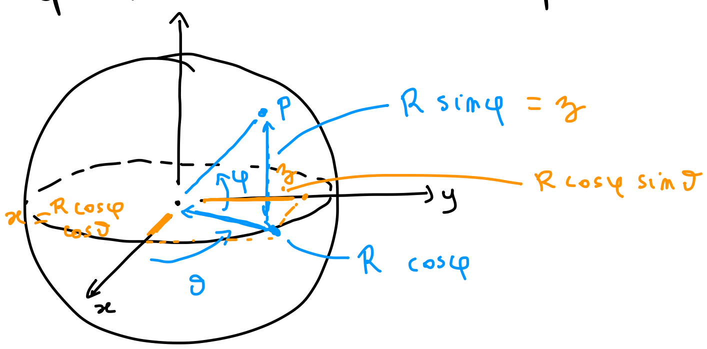
$$\frac{\partial \rho}{\partial u} \wedge \frac{\partial \rho}{\partial v}$$

Si la paramétrisation est non dégénérée au pt P

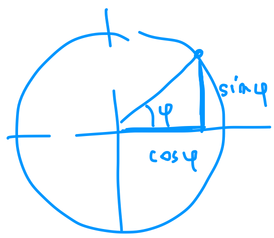
Paramétrisation dégénérée en $P = \rho(u_0, v_0)$

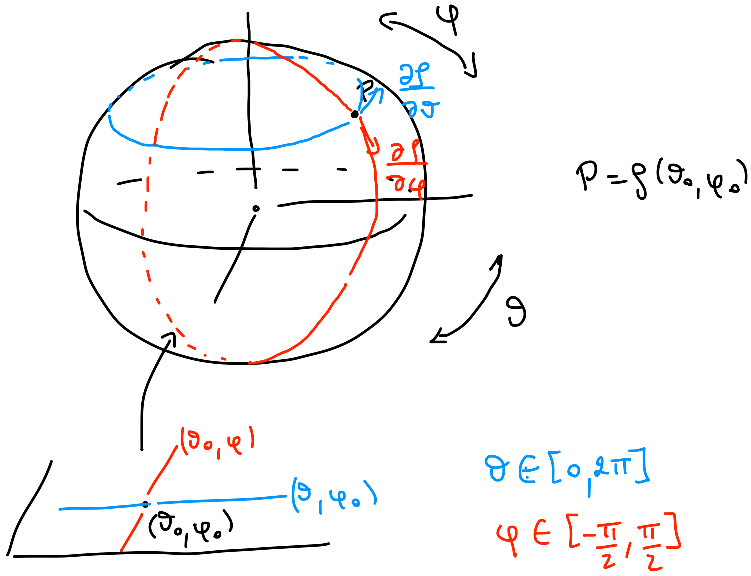
si
$$\frac{\partial \rho}{\partial u}(u_0, v_0) \wedge \frac{\partial \rho}{\partial v}(u_0, v_0) = \vec{0}$$

ex: paramétrisation de la sphère (rayon R)



$$\rho(\vartheta, \varphi) = \begin{pmatrix} R \cos\varphi \cos\vartheta \\ R \cos\varphi \sin\vartheta \\ R \sin\varphi \end{pmatrix}$$





$$\frac{\partial \rho}{\partial \vartheta} = \begin{pmatrix} -R \cos\varphi \sin\vartheta \\ R \cos\varphi \cos\vartheta \\ 0 \end{pmatrix}$$

$\vartheta \in [0, 2\pi]$

$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\frac{\partial \rho}{\partial \varphi} = \begin{pmatrix} -R \sin\varphi \cos\vartheta \\ -R \sin\varphi \sin\vartheta \\ R \cos\varphi \end{pmatrix}$$

Pôle Nord $\longleftrightarrow \left(\vartheta, \frac{\pi}{2}\right)$ $\forall \vartheta$

" Sud $\longleftrightarrow \left(\vartheta, -\frac{\pi}{2}\right)$

Tangentes au pôle Nord :

$$\frac{\partial \rho}{\partial \theta} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0} \qquad \frac{\partial \rho}{\partial \varphi} = \begin{pmatrix} -R\cos\theta \\ -R\sin\theta \\ 0 \end{pmatrix}$$

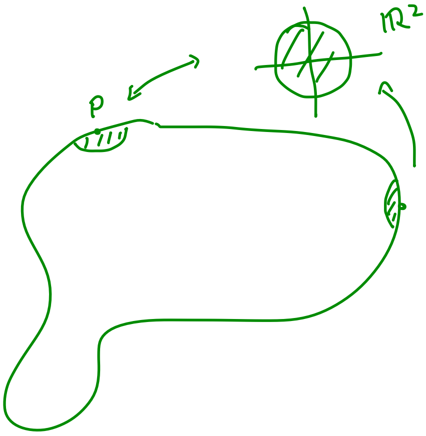
Paramétrisation dégénérée

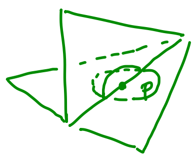
<u>Nota</u> (manifold)

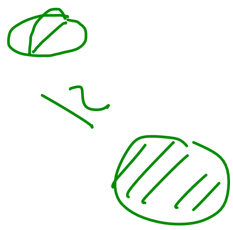
Variété (de dimension 2) $\longrightarrow$ généralisation maths des surfaces (paramétriques)

Espace $\mathcal{M}$ tq

$\forall P \in \mathcal{M}$ P a un voisinage $\underset{\text{isomorphe}}{\simeq}$ $\mathcal{D}^2$ (disque de dim 2)







2) Courbures

a) (R)appel sur les formes quadratiques

Matrice A $\longleftrightarrow$ appli. linéaire $f(x)$ 90% $A'X$

d°1

dualité

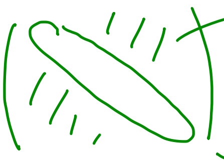
Matrice Ω $\longleftrightarrow$ forme quadratique ${}^tX.\Omega.X$

d°2 coefs $\underbrace{x, y, z}_{\mathbb{R}^3}$

$xy + yx = 2xy$

$x^2 + 2xy - y^2 + xz$

$$\hookrightarrow \begin{array}{c|ccc} & x & y & z \\ \hline x & 1 & 1 & 1/2 \\ y & 1 & -1 & 0 \\ z & 1/2 & 0 & 0 \end{array}$$



termes croisés répartis symétriquement

coefs des termes carrés

e) Première forme fondamentale $\longleftrightarrow$ dérivées premières

Matrice 2×2 au pt $P = f(u_0, v_0)$ $\quad \dfrac{\partial f}{\partial u} \quad \dfrac{\partial f}{\partial v}$

mesure le défaut de cette base à être orthogonale / orthonormale

$$\boxed{I_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix}}$$

$$E = \frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial u} = \left\| \frac{\partial f}{\partial u} \right\|^2$$

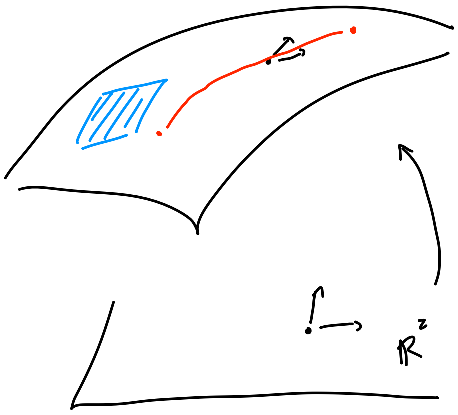
$$F = \frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v}$$

$$G = \frac{\partial f}{\partial v} \cdot \frac{\partial f}{\partial v} = \left\| \frac{\partial f}{\partial v} \right\|^2$$

si la base $\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)$ du plan tangent est orthonormale

$$I_p = Id$$

$\rightsquigarrow$ sert à mesurer les longueurs et aires sur la courbe



e) Deuxième forme fondamentale

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\frac{\partial^2 f}{\partial u^2} \quad \frac{\partial^2 f}{\partial u \partial v} = \frac{\partial^2 f}{\partial v \partial u} \quad \frac{\partial^2 f}{\partial v^2} \quad \in \mathbb{R}^3$$

si param. non dégénérée

Normale au pt P :

$$n_p = \frac{\frac{\partial f}{\partial u} \wedge \frac{\partial f}{\partial v}}{\left\| \frac{\partial f}{\partial u} \wedge \frac{\partial f}{\partial v} \right\|}$$

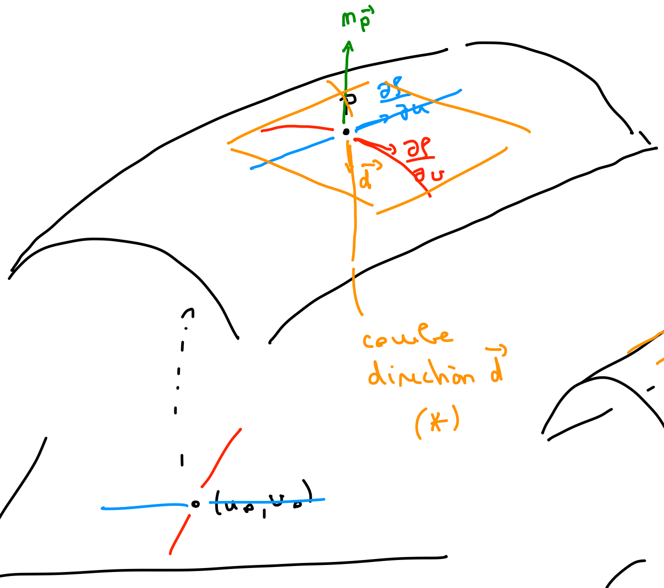
$$\boxed{\mathrm{II}_p = \begin{pmatrix} \ell & m \\ m & n \end{pmatrix}}$$

$$\begin{cases} \ell = \dfrac{\partial^2 \rho}{\partial u^2} \cdot n_p \\[2mm] m = \dfrac{\partial^2 \rho}{\partial u \partial v} \cdot n_p \\[2mm] n = \dfrac{\partial^2 \rho}{\partial v^2} \end{cases}$$

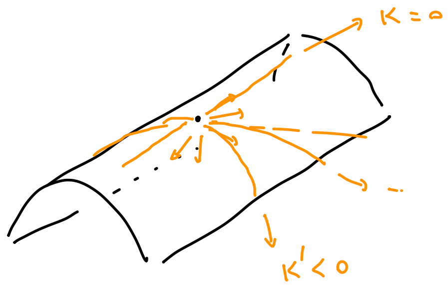
c) Courbures normales / courbures

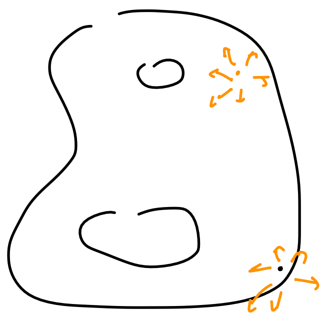
Courbure → une par direction ? $\vec{d}$

→ vecteurs tangents



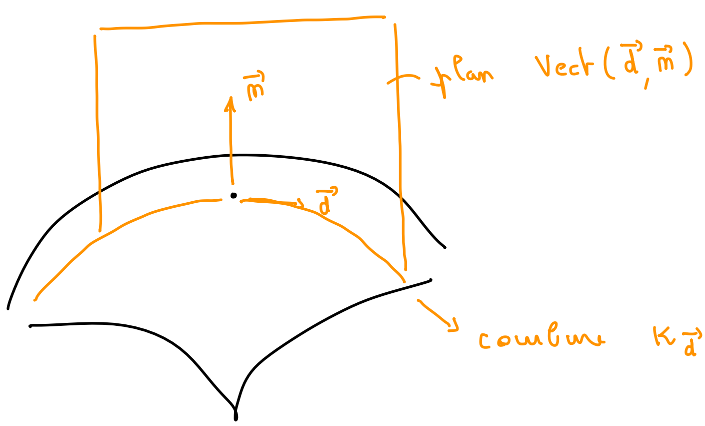
courbe direction $\vec{d}$ (*)





(*) Intersection → surface

→ plan normal de direction $\vec{d}$ (contient $\vec{d}, \vec{n}$)



plan Vect$(\vec{d}, \vec{n})$

courbure $\kappa_{\vec{d}}$: courbure normale ds la direction $\vec{d}$

Les courbures de la surf. en P

↳ fonction : $\vec{d} \longmapsto K_{\vec{d}}$

forme quadratique — poly. d° 2 exactement

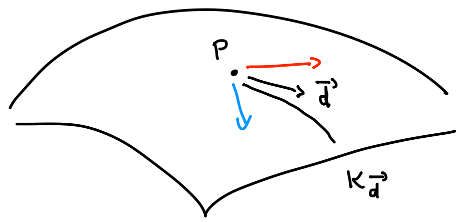
sa matrice 2×2 (dans la base du plan tangent)

$\mathcal{K}_P$ tenseur de courbure

$$\boxed{\mathcal{K}_P = I_P^{-1} \times II_P}$$

matrice 2×2

rem : symétrique



$$K_{\vec{d}} = \vec{d}^{\,t} \times \mathcal{K}_P \times \vec{d}$$

directions dans le plan tangent de base $\left(\frac{\partial S}{\partial u}, \frac{\partial S}{\partial v}\right)$

2×2

$\mathcal{K}_P$ est diagonalisable en base orthonormale

⇝ $\vec{d}_1, \vec{d}_2$: vecteurs propres) directions principales

⇝ K_1, K_2 : valeurs propres associées) courbures principales

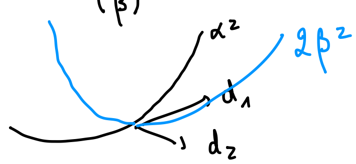
Dans la base $(\vec{d}_1, \vec{d}_2)$ la matrice $\mathcal{K}_P$ devient :

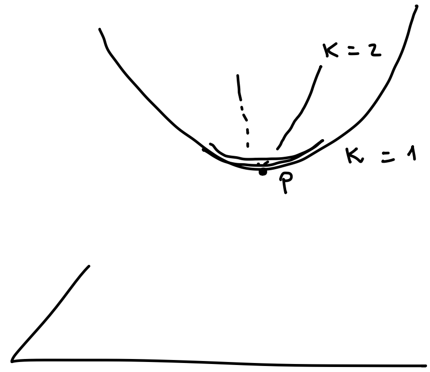
$$\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \longleftrightarrow K_{\vec{d}} = {}^t\vec{d} \times \mathcal{K}_P \times \vec{d}$$

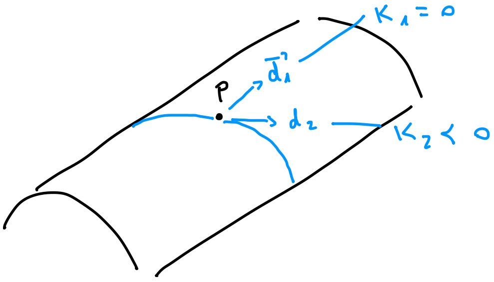
$$\vec{d} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad = K_1 \alpha^2 + K_2 \beta^2$$

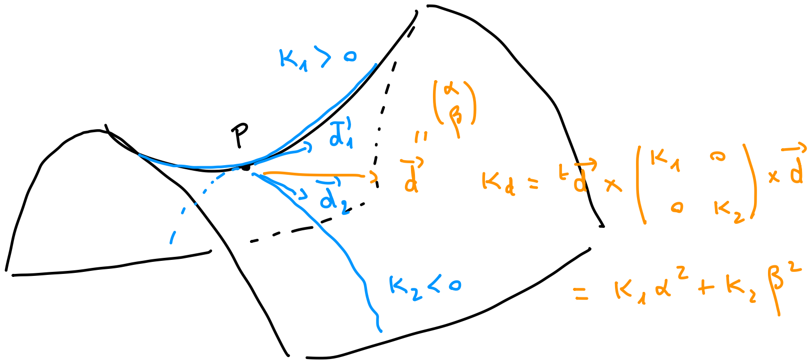
ex : $(K_1, K_2) = (1, 2)$

$$K_{\vec{d}} = \alpha^2 + 2\beta^2$$









Courbure
- gaussien $G = K_1 K_2$ (moy. géométrique $\sqrt{K_1 K_2}$)
- moyenne $H = \frac{K_1 + K_2}{2}$

2 moyennes (arith / géom) de K_1, K_2