Measuring and Representing Holes in Discrete Objects

Aldo Gonzalez-Lorenzo

Aix-Marseille Université - CNRS, LIS UMR 7020

March 25, 2019

Outline

1 Introduction

2 Background

3 Measuring Holes

4 Representing Holes

5 Applications

6 Conclusion

Outline

1 Introduction

2 Background

3 Measuring Holes

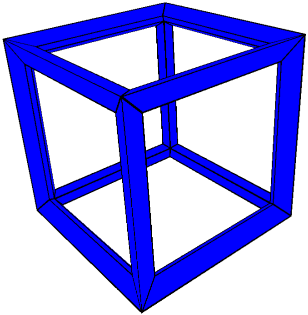
4 Representing Holes

5 Applications

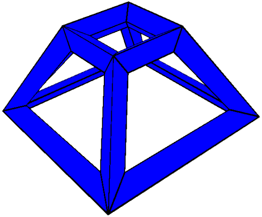
6 Conclusion

Holes $\simeq$ homology

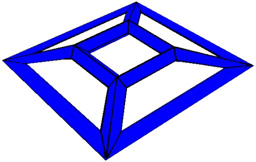
Holes $\simeq$ homology



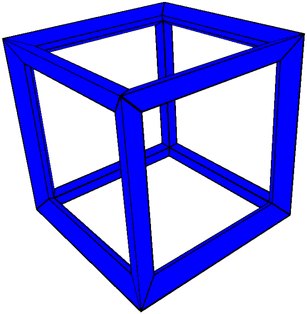
Holes $\simeq$ homology



Holes $\simeq$ homology



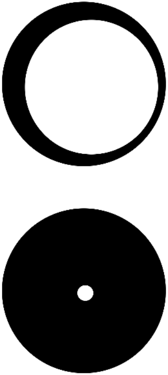
Holes $\simeq$ homology



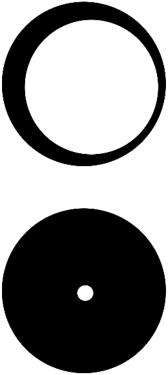
- We can know **how many** holes there are in an object
- We cannot know **where** or **how** they are

Size of a hole

The 1st one is *bigger* than the 2nd one

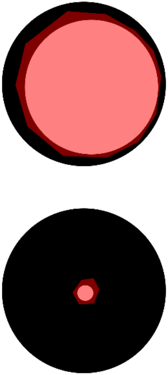


The 2nd one is *thicker* than the 1st one

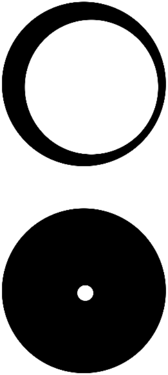


Size of a hole

The 1st one is *bigger* than the 2nd one

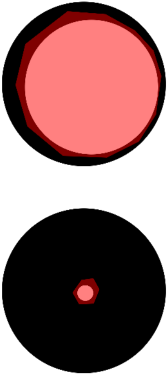


The 2nd one is *thicker* than the 1st one

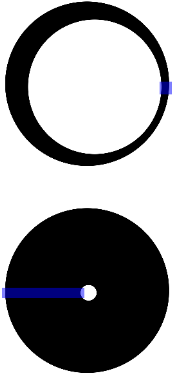


Size of a hole

The 1st one is *bigger* than the 2nd one

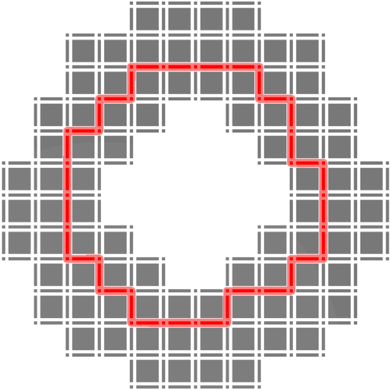


The 2nd one is *thicker* than the 1st one

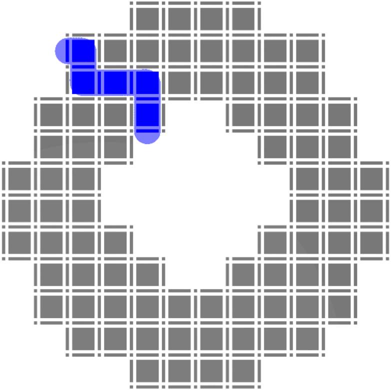


Representing a hole

Homology

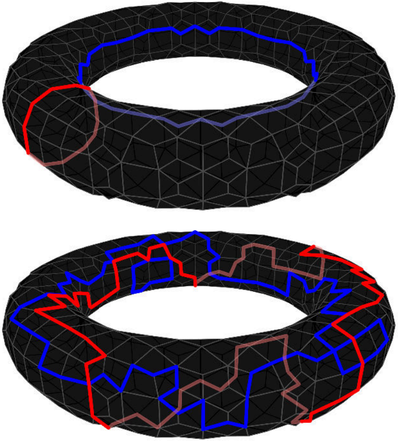


Cohomology



Representing a hole

Do homology generators really represent holes?



Geometry	+	Topology
↓		↓
Signed distance transform		Persistent homology

Outline

1 Introduction

2 Background
- Digital Geometry
- Cubical Complexes
- Homology
- Persistent Homology

3 Measuring Holes

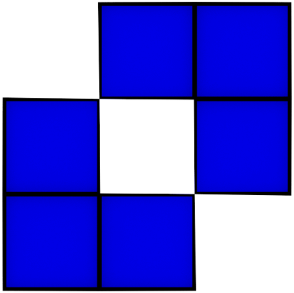
4 Representing Holes

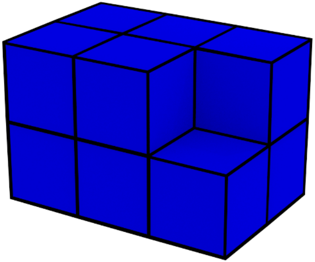
5 Applications

6 Conclusion

Discrete object

A nD discrete object is a subset of $\mathbb{Z}^n$

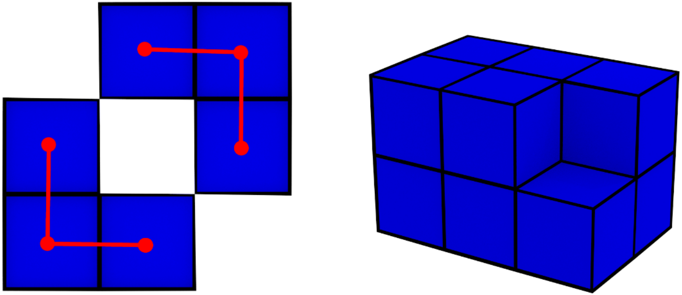




Connectivity relation: $2n$ or the $(3^n - 1)$-connectivity.

Discrete object

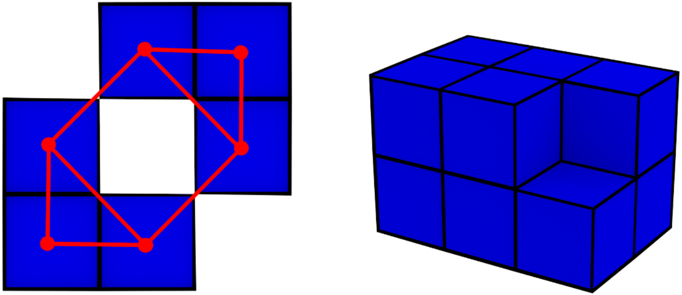
A nD discrete object is a subset of $\mathbb{Z}^n$



Connectivity relation: $2n$ or the $(3^n - 1)$-connectivity.

Discrete object

A nD discrete object is a subset of $\mathbb{Z}^n$



Connectivity relation: $2n$ or the $(3^n - 1)$-connectivity.

Signed distance transform

Let O be a discrete object,

$$sdt_O(x) = \begin{cases} -d(x, O^c) & \text{if } x \in O \\ d(x, O) & \text{if } x \notin O \end{cases}$$

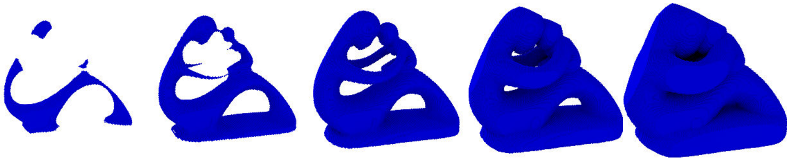
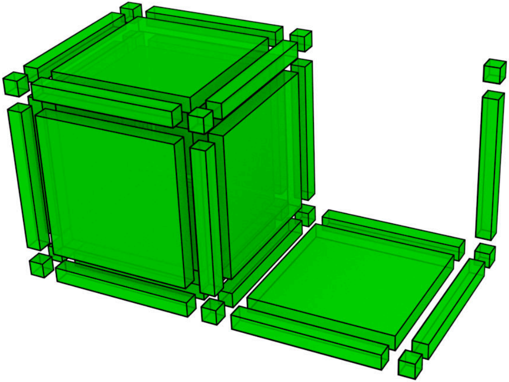


Figure: Sublevel sets of the signed distance transform

Cubical complex

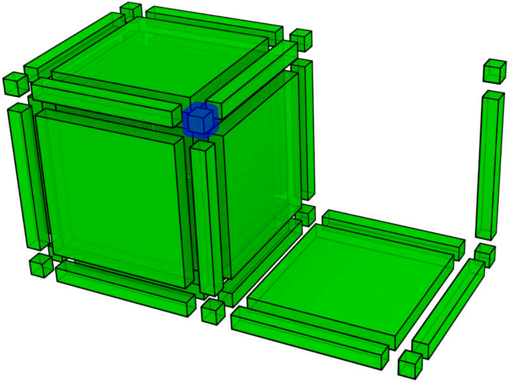
Union[1] of points, edges, squares, cubes, ... (cubes)



[1] with some conditions

Cubical complex

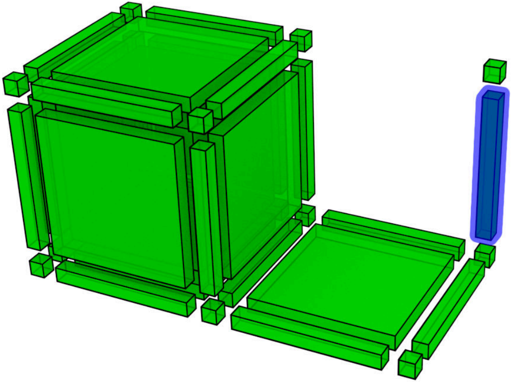
Union[1] of **points**, edges, squares, cubes, ... (cubes)



[1] with some conditions

Cubical complex

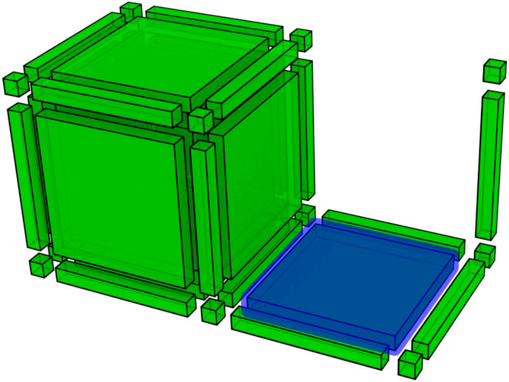
Union[1] of points, **edges**, squares, cubes, ... (cubes)



[1]with some conditions

Cubical complex

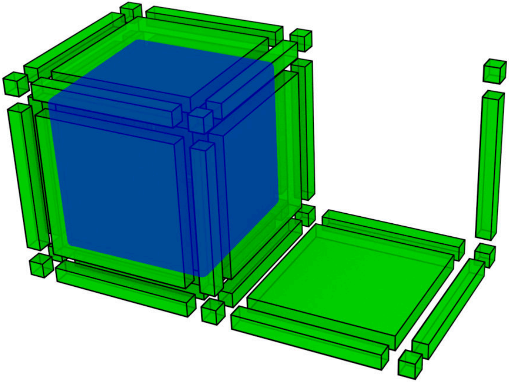
Union[1] of points, edges, **squares**, cubes, ... (cubes)



[1]with some conditions

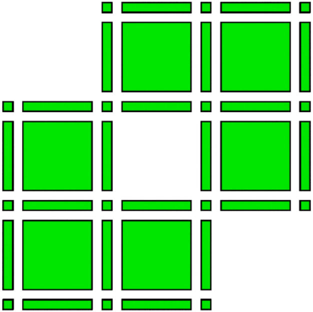
Cubical complex

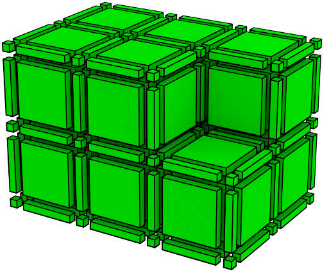
Union[1] of points, edges, squares, **cubes**, ... (cubes)



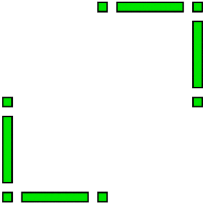
[1] with some conditions

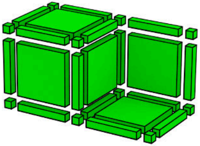
Discrete object $\longrightarrow$ cubical complex ($(3^n - 1)$-connectivity)





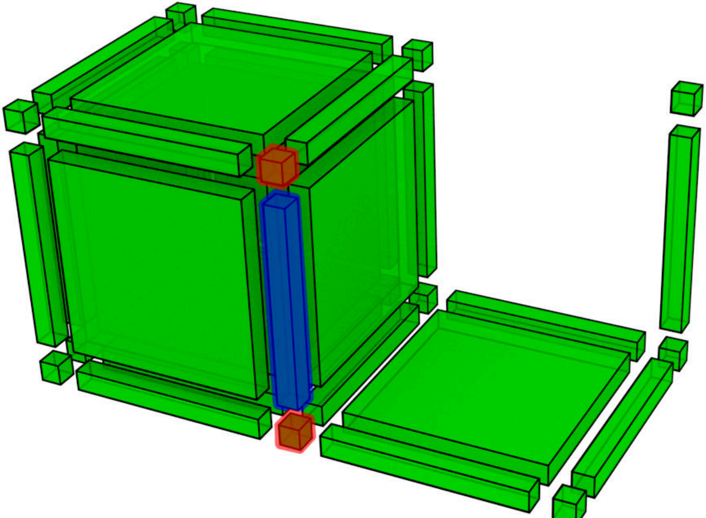
Discrete object $\longrightarrow$ cubical complex ($2n$-connectivity)





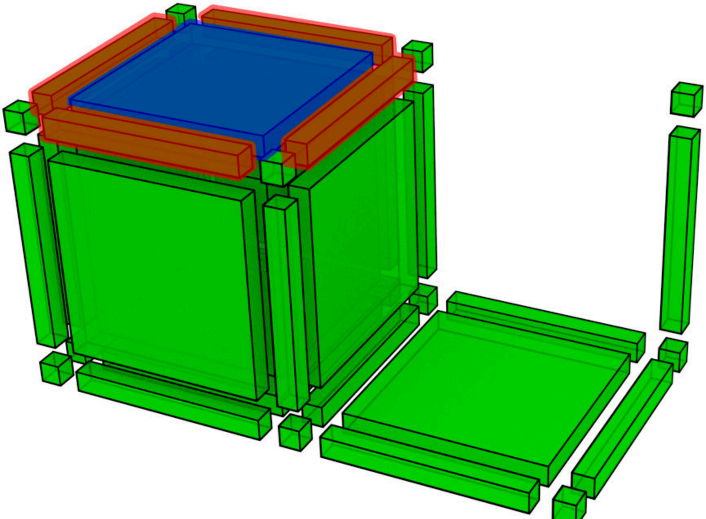
Blue: 1-cube

Red: its boundary (faces)



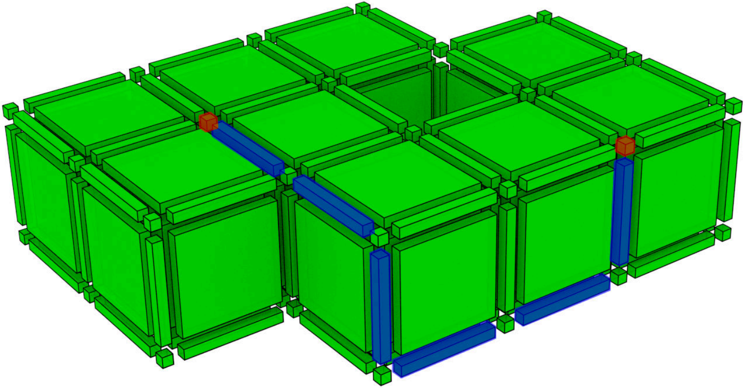
Blue: 2-cube

Red: its boundary (faces)



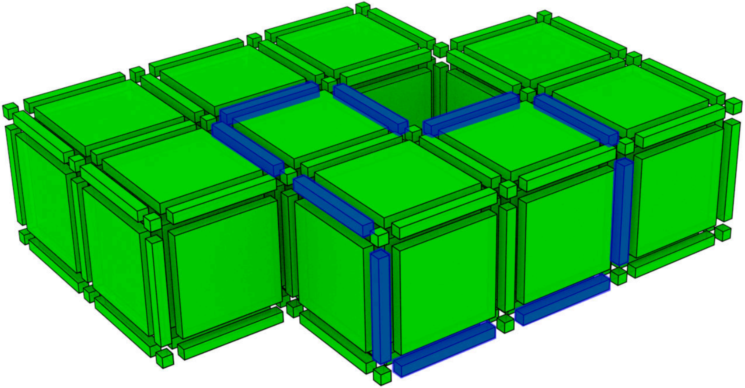
Blue: 1-chain

Red: its boundary



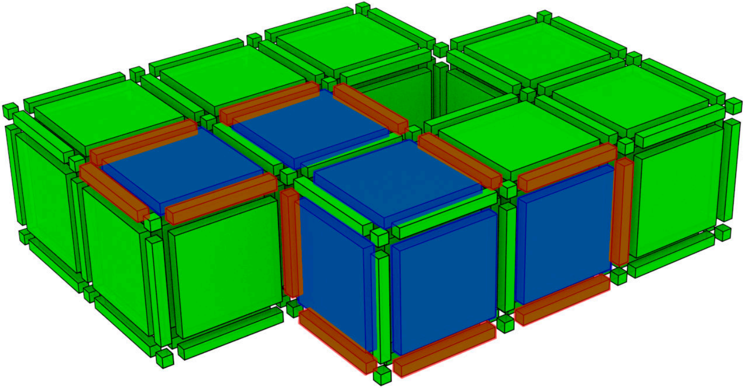
Blue: 1-chain (1-cycle)

Red: its boundary ($= \emptyset$)



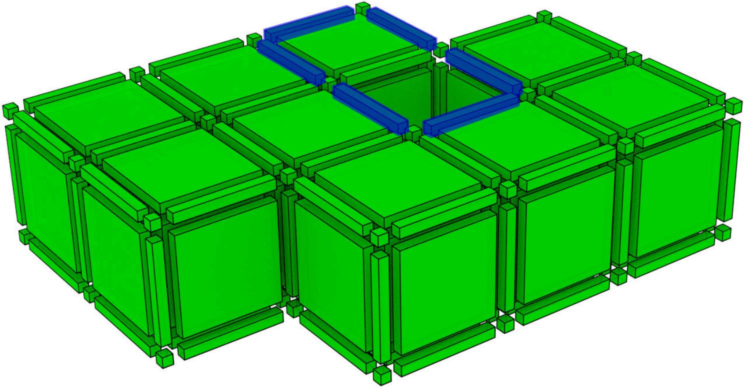
Blue: 2-chain

Red: its boundary (1-cycle)



Blue: 1-chain (1-cycle, but not boundary)

Red: its boundary ($= \emptyset$)



- K cubical complex
- Chain complex of K

$$\cdots C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

where $d_q d_{q+1} = 0 \Rightarrow \mathrm{im}(d_{q+1}) \subset \ker(d_q)$

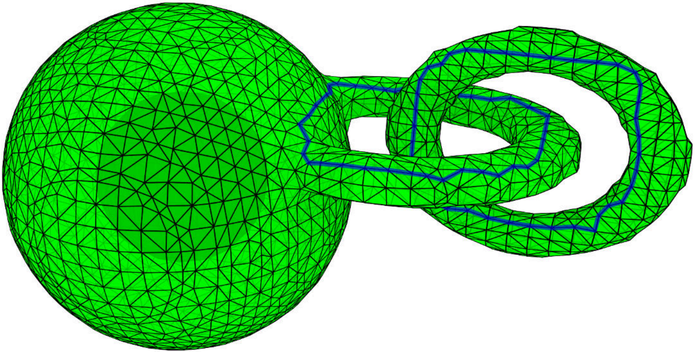
- q-dimensional homology group
  $H_q(K) := \ker(d_q)/\,\mathrm{im}(d_{q+1})$[2] $= (\mathbb{F}_2)^{\beta_q}$
- q-dimensional Betti number: β_q

[2] $\forall x, y \in \ker(d_q), x \sim y \Leftrightarrow x + y \in \mathrm{im}(d_{q+1})$

- $\beta_0 = \#$ connected components (0-holes)
- $\beta_1 = \#$ tunnels or handles (1-holes)
- $\beta_2 = \#$ cavities (2-holes)

Betti numbers are

- Topological invariants $\rightarrow$ classification
- Shape descriptors $\rightarrow$ understanding

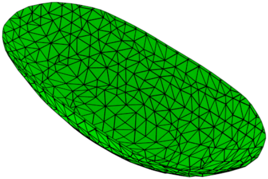


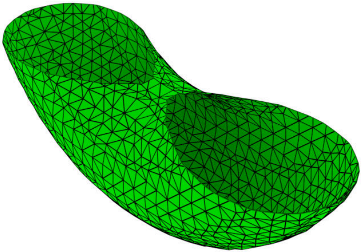
$\beta_0 = 2,\ \beta_1 = 2,\ \beta_2 = 1,\ \beta_3 = 0,\ \ldots$

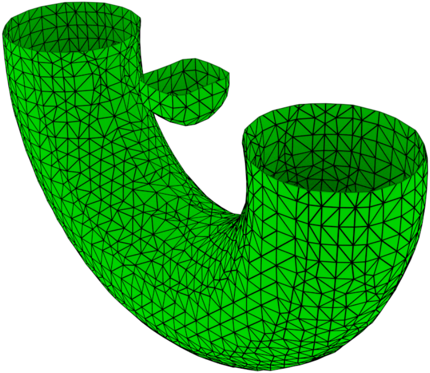
- Filtration F: $K_1 \subset K_2 \subset K_3 \subset \cdots$

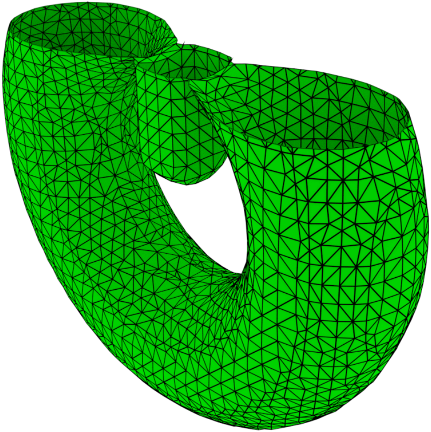
$$\begin{array}{ccccccc} K_1 & \xrightarrow{\iota} & K_2 & \xrightarrow{\iota} & K_3 & \xrightarrow{\iota} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ H(K_1) & \xrightarrow{\iota_*} & H(K_2) & \xrightarrow{\iota_*} & H(K_3) & \xrightarrow{\iota_*} & \cdots \end{array}$$

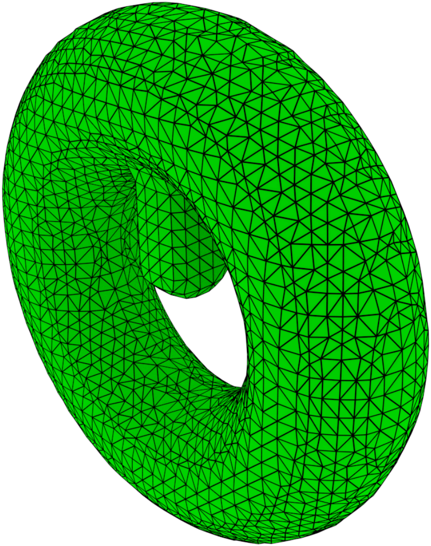
- $\beta_{i,j} = \dim(\iota : H(K_i) \to H(K_j))$
  number of holes in K_i still in K_j
- $\mu_{i,j} = \beta_{i,j} - \beta_{i,j+1} - \beta_{i-1,j} + \beta_{i-1,j+1}$
  number of holes born in K_i and dying in K_j







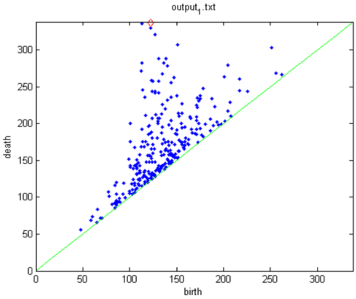




Persistence pairs

$PD(F) = \{(i,j) \text{ with multiplicity } \mu_{i,j}\}$

We represent $PD(F)$ with a *persistence diagram*.



Outline

1 Introduction

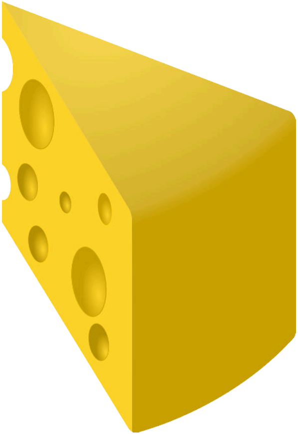
2 Background

3 Measuring Holes

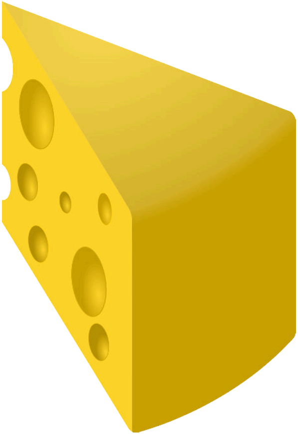
4 Representing Holes

5 Applications

6 Conclusion



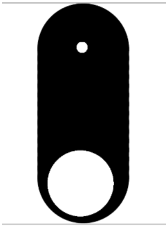
- $\beta_2 = 10$
- "Full" of holes?
- All holes have similar size?



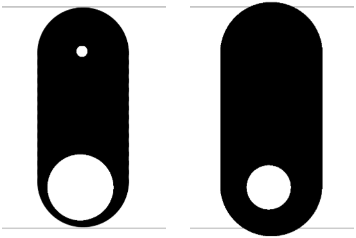
- $\beta_2 = 10$
- "Full" of holes?
- All holes have similar size?

Dilate... and erode

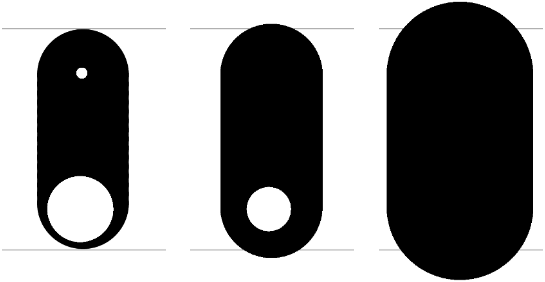
Persistent homology with signed distance transform



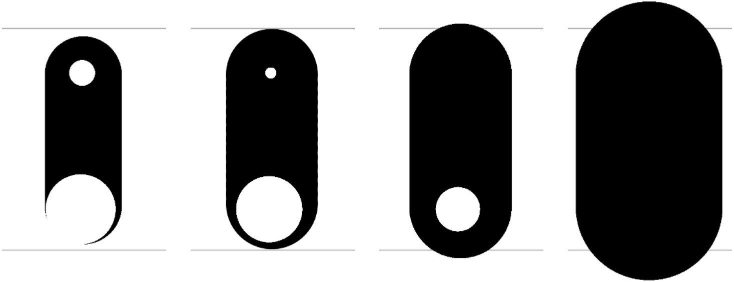
Persistent homology with signed distance transform



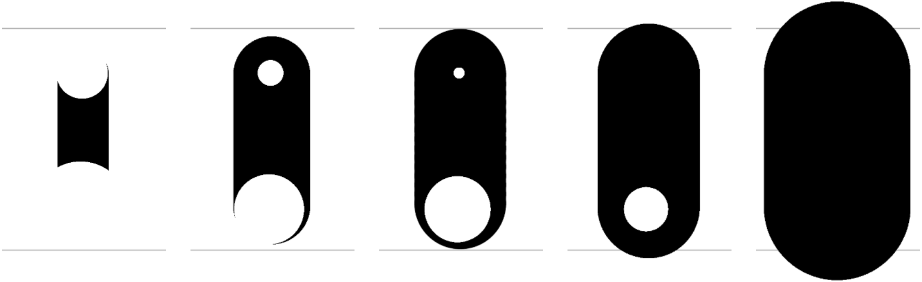
Persistent homology with signed distance transform



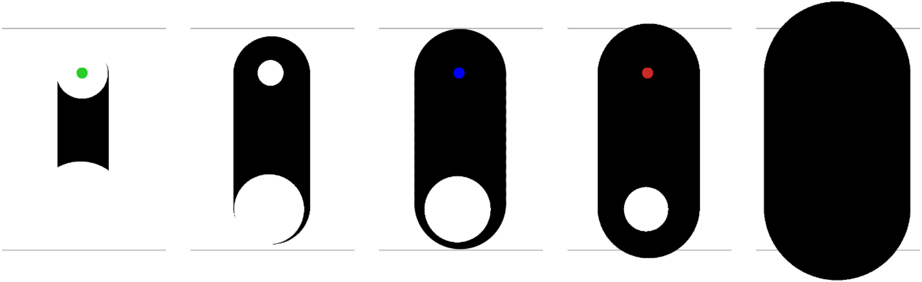
Persistent homology with signed distance transform



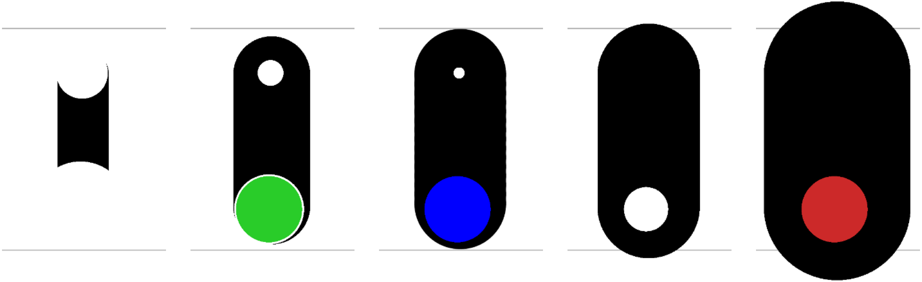
Persistent homology with signed distance transform



Persistent homology with signed distance transform



Persistent homology with signed distance transform



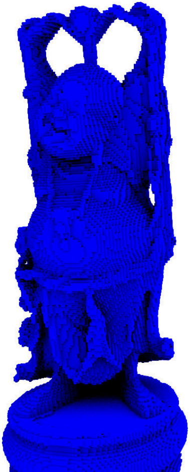
Thickness and breadth

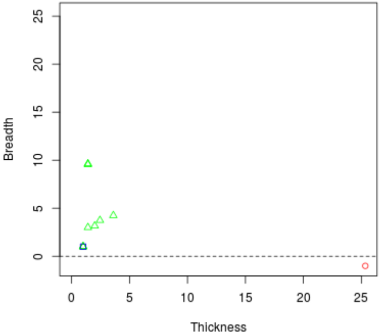
Let O be a discrete object and F the filtration defined by the sublevel sets of its signed distance transform.
Let $TB(O) = \{(-x, y) \in PD(F) \mid x \leq 0, y \geq 0\}$.
Its intervals are the *thickness-breadth* pairs of O.

- One thickness-breadth pair (t, b) for each hole of O
- t is the *thickness* of the hole and b, its *breadth*

Thickness-breadth diagram

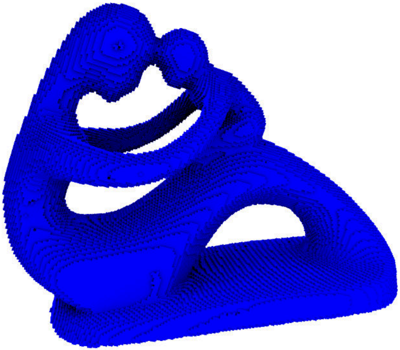
Thickness-breadth pairs can be represented like persistence diagrams

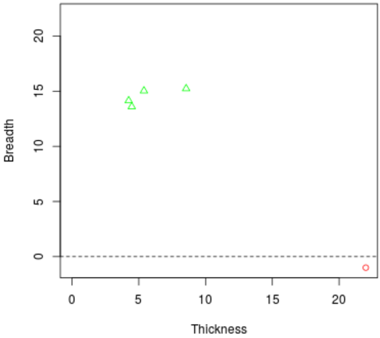




Thickness-breadth diagram

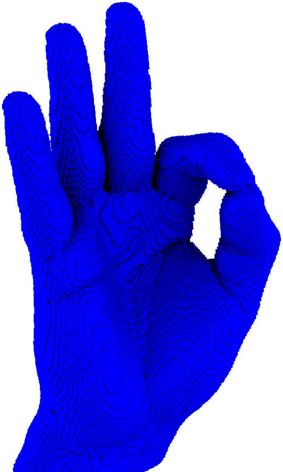
Thickness-breadth pairs can be represented like persistence diagrams

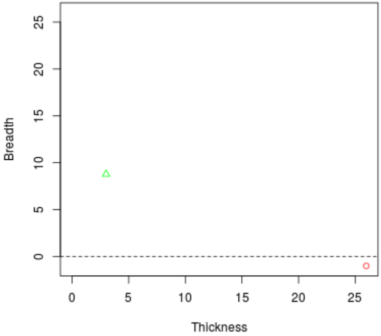




Thickness-breadth diagram

Thickness-breadth pairs can be represented like persistence diagrams





Theorem

Let X and Y be two 3D discrete objects. Let us call

$$\delta = d_H(X, Y) + d_H(\mathbb{Z}^3 \setminus X, \mathbb{Z}^3 \setminus Y) + 2\sqrt{3}$$

Thus, for every thickness-breadth pair $p_X = (x, y)$ of X such that $x, y > \delta$, there exists another thickness-breadth pair $p_Y = (x', y')$ of Y such that

$$||p_X - p_Y||_\infty \leq \delta$$

Outline

1 Introduction

2 Background

3 Measuring Holes

4 Representing Holes

5 Applications

6 Conclusion

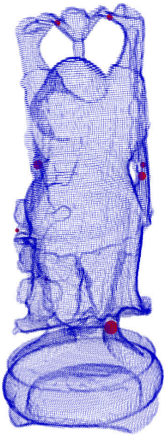
Sketch of persistent homology computation:

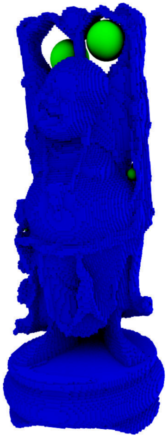
- Sort cells according to the filtration
- For each cell, associate it with one of the previous ones
- Each of these pairs makes a persistence pair

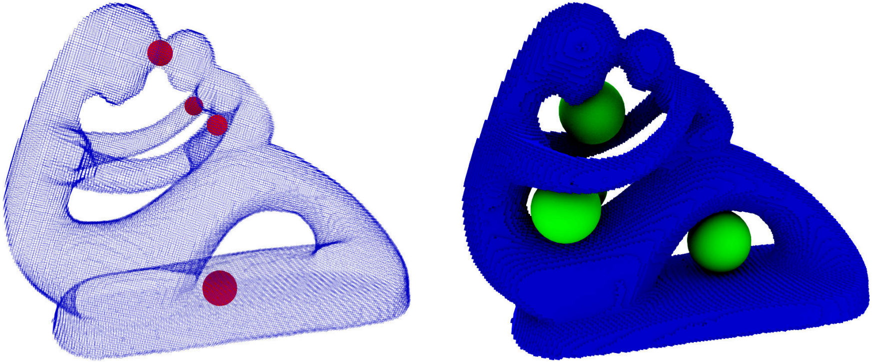
Thickness and breadth ball

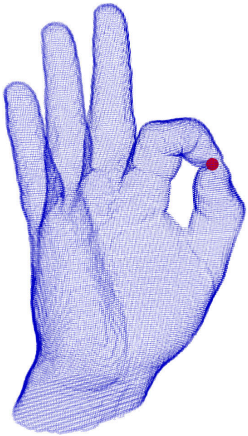
Let (t, b) be a TB-pair and (σ, τ) its pair of cells

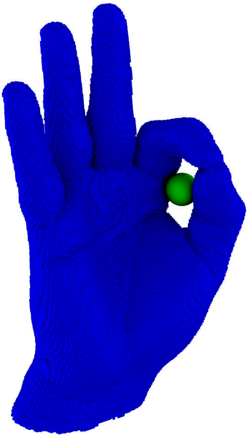
- The *thickness ball* of (t, b) is the ball of radius t centered at σ
- The *breadth ball* of (t, b) is the ball of radius b centered at τ











We can identify each hole by

- Its thickness-breadth pair (unique)
- The center of its thickness ball (non unique)
- The center of its breadth ball (non unique)

Outline

1 Introduction

2 Background

3 Measuring Holes

4 Representing Holes

5 Applications

6 Conclusion

Given a discrete object O and a TB pair (t, b)

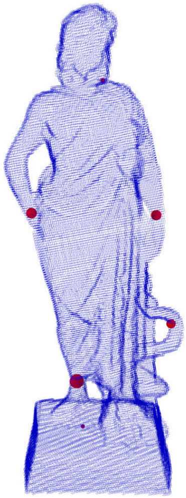
- Open the hole: remove voxels from O
- Close the hole: add voxels to O to remove the hole
- Compute a homology generator
- Compute a cohomology generator

Why?

Topological correction

Example

1. Scan an object, segmentate it and compute its TB diagram.
2. Identify “wrong” holes using TB pairs and balls.
3. Open or close them.



Extract relation between holes (1/2)

Example

1. For each hole, close it and see which holes vanish.
2. Represent this with a graph.
3. Compare objects using these graphs.

Extract relation between holes (2/2)

Example

1. For each hole, compute its homology generator.
2. Transform these generators into discrete objects.
3. Extract the relation between these holes.

Outline

1 Introduction

2 Background

3 Measuring Holes

4 Representing Holes

5 Applications

6 Conclusion

Conclusion:

- Topological-geometrical signature of objects.
- Robust to noise $\rightarrow$ suitable for real applications.
- Alternative visualization of holes.
- Heuristics for minimal openings, closings and (co)homology generators.

Conclusion:

- Topological-geometrical signature of objects.
- Robust to noise $\rightarrow$ suitable for real applications.
- Alternative visualization of holes.
- Heuristics for minimal openings, closings and (co)homology generators.

Thanks! Questions?