Measuring and Representing Holes in Discrete Objects

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Outline

1. Introduction
2. Background
3. Measuring Holes
4. Representing Holes
5. Applications
6. Conclusion
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6. Conclusion
Holes $\simeq$ homology
Holes $\simeq$ homology
Holes \sim \text{homology}
Holes $\simeq$ homology
Holes $\cong$ homology

- We can know **how many** holes there are in an object
- We cannot know **where** or **how** they are
Size of a hole

The 1st one is *bigger* than the 2nd one

The 2nd one is *thicker* than the 1st one
Size of a hole

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Size of a hole

The 1st one is *bigger* than the 2nd one

The 2nd one is *thicker* than the 1st one
Representing a hole

Homology

Cohomology
Representing a hole

Do homology generators really represent holes?
Measuring and Representing Holes in Discrete Objects

Introduction

Geometry + Topology

↓ ↓

Signed distance transform Persistent homology
Outline

1. Introduction

2. Background
   - Digital Geometry
   - Cubical Complexes
   - Homology
   - Persistent Homology

3. Measuring Holes

4. Representing Holes

5. Applications

6. Conclusion
A $n$D discrete object is a subset of $\mathbb{Z}^n$.

Connectivity relation: $2n$ or the $(3^n - 1)$-connectivity.
Discrete object

A $n$D discrete object is a subset of $\mathbb{Z}^n$

Connectivity relation: $2n$ or the $(3^n - 1)$-connectivity.
A $n$D discrete object is a subset of $\mathbb{Z}^n$.

Connectivity relation: $2n$ or the $(3^n - 1)$-connectivity.
Signed distance transform

Let $O$ be a discrete object,

$$sdt_O(x) = \begin{cases} -d(x, O^c) & \text{if } x \in O \\ d(x, O) & \text{if } x \notin O \end{cases}$$

**Figure:** Sublevel sets of the signed distance transform
Cubical complex

Union\(^1\) of points, edges, squares, cubes, ... (cubes)

\(^1\) with some conditions
Cubical complex

Union$^1$ of points, edges, squares, cubes, ... (cubes)

$^1$with some conditions
Cubical complex

Union\(^1\) of points, **edges**, squares, cubes, ... (cubes)

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Cubical complex

Union\(^1\) of points, edges, squares, **cubes**, ... (cubes)

\(^{1}\text{with some conditions}\)
Discrete object $\rightarrow$ cubical complex ($(3^n - 1)$-connectivity)
Discrete object $\longrightarrow$ cubical complex ($2n$-connectivity)
**Blue:** 1-cube

**Red:** its boundary (faces)
Measuring and Representing Holes in Discrete Objects

- **Background**
- **Homology**

**Blue:** 2-cube
**Red:** its boundary (faces)
Blue: 1-chain
Red: its boundary
Blue: 1-chain (1-cycle)
Red: its boundary (= ∅)
Blue: 2-chain
Red: its boundary (1-cycle)
**Background**

**Homology**

**Blue**: 1-chain (1-cycle, but not boundary)

**Red**: its boundary \((= \emptyset)\)
- \( K \) cubical complex
- Chain complex of \( K \)

\[
\cdots \xrightarrow{d_3} C_3 \xrightarrow{d_2} C_2 \xrightarrow{d_1} C_1 \xrightarrow{d_0} C_0 \xrightarrow{d_0} 0
\]

where \( d_q d_{q+1} = 0 \Rightarrow \text{im}(d_{q+1}) \subset \ker(d_q) \)

- \( q \)-dimensional homology group
  \[ H_q(K) := \ker(d_q)/\text{im}(d_{q+1})^2 = (\mathbb{F}_2)^{\beta_q} \]
- \( q \)-dimensional Betti number: \( \beta_q \)

\[\forall x, y \in \ker(d_q), x \sim y \iff x + y \in \text{im}(d_{q+1})\]
Betti numbers are

- $\beta_0 = \#$ connected components (0-holes)
- $\beta_1 = \#$ tunnels or handles (1-holes)
- $\beta_2 = \#$ cavities (2-holes)

Betti numbers are

- Topological invariants $\rightarrow$ classification
- Shape descriptors $\rightarrow$ understanding
\[ \beta_0 = 2, \beta_1 = 2, \beta_2 = 1, \beta_3 = 0, \ldots \]
Filtration $F$: $K_1 \subset K_2 \subset K_3 \subset \cdots$

\[
\begin{array}{cccc}
K_1 & \xrightarrow{\iota} & K_2 & \xrightarrow{\iota} & K_3 & \xrightarrow{\iota} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
H(K_1) & \xrightarrow{\iota^*} & H(K_2) & \xrightarrow{\iota^*} & H(K_3) & \xrightarrow{\iota^*} & \cdots \\
\end{array}
\]

$\beta_{i,j} = \dim(\iota : H(K_i) \to H(K_j))$

- number of holes in $K_i$ still in $K_j$

$\mu_{i,j} = \beta_{i,j} - \beta_{i,j+1} - \beta_{i-1,j} + \beta_{i-1,j+1}$

- number of holes born in $K_i$ and dying in $K_j$
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- Background
- Persistent Homology
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Background

Persistent Homology
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Background

Persistent Homology
### Persistence pairs

$$PD(F) = \{(i, j) \text{ with multiplicity } \mu_{i,j}\}$$

We represent $PD(F)$ with a persistence diagram.
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Measuring Holes

- $\beta_2 = 10$
- “Full” of holes?
- All holes have similar size?
Measuring Holes

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- “Full” of holes?
- All holes have similar size?

Dilate... and erode
Persistent homology with signed distance transform
Persistent homology with signed distance transform
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Persistent homology with signed distance transform
Persistent homology with signed distance transform
Thickness and breadth

Let $O$ be a discrete object and $F$ the filtration defined by the sublevel sets of its signed distance transform. Let $TB(O) = \{(−x, y) \in PD(F) \mid x \leq 0, y \geq 0\}$. Its intervals are the thickness-breadth pairs of $O$.

- One thickness-breadth pair $(t, b)$ for each hole of $O$
- $t$ is the thickness of the hole and $b$, its breadth
Thickness-breadth diagram

Thickness-breadth pairs can be represented like persistence diagrams
Thickneess-breadth diagram

Thickneess-breadth pairs can be represented like persistence diagrams
Thickness-breadth diagram

Thickness-breadth pairs can be represented like persistence diagrams
Theorem

Let $X$ and $Y$ be two 3D discrete objects. Let us call

$$\delta = d_H(X, Y) + d_H(\mathbb{Z}^3 \setminus X, \mathbb{Z}^3 \setminus Y) + 2\sqrt{3}$$

Thus, for every thickness-breadth pair $p_X = (x, y)$ of $X$ such that $x, y > \delta$, there exists another thickness-breadth pair $p_Y = (x', y')$ of $Y$ such that

$$\|p_X - p_Y\|_\infty \leq \delta$$
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Sketch of persistent homology computation:
- Sort cells according to the filtration
- For each cell, associate it with one of the previous ones
- Each of these pairs makes a persistence pair

**Thickness and breadth ball**

Let \((t, b)\) be a TB-pair and \((\sigma, \tau)\) its pair of cells
- The *thickness ball* of \((t, b)\) is the ball of radius \(t\) centered at \(\sigma\)
- The *breadth ball* of \((t, b)\) is the ball of radius \(b\) centered at \(\tau\)
Representing Holes

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Representing Holes

[Images of two complex 3D objects with different color representations showing holes]
We can identify each hole by

- Its thickness-breadth pair (unique)
- The center of its thickness ball (non unique)
- The center of its breadth ball (non unique)
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Given a discrete object $O$ and a TB pair $(t, b)$

- Open the hole: remove voxels from $O$
- Close the hole: add voxels to $O$ to remove the hole
- Compute a homology generator
- Compute a cohomology generator

Why?
Topological correction

Example

1. Scan an object, segmentate it and compute its TB diagram.
2. Identify “wrong” holes using TB pairs and balls.
3. Open or close them.
Extract relation between holes (1/2)

Example

1. For each hole, close it and see which holes vanish.
2. Represent this with a graph.
3. Compare objects using these graphs.
Extract relation between holes (2/2)

Example

1. For each hole, compute its homology generator.
2. Transform these generators into discrete objects.
3. Extract the relation between these holes.
Conclusion:

- Topological-geometrical signature of objects.
- Robust to noise → suitable for real applications.
- Alternative visualization of holes.
- Heuristics for minimal openings, closings and (co)homology generators.
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Thanks! Questions?