Computational Homology Applied to Discrete Objects

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November 24, 2016

Structure

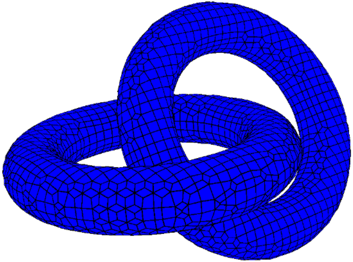
1 Introduction and Preliminaries

2 The Homological Discrete Vector Field

3 Fast Computation of Betti Numbers on 3D Cubical Complexes

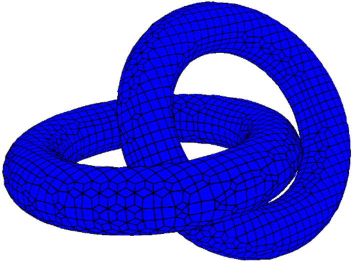
4 Measuring Holes

5 Conclusion



Geometry:

- Volume
- Diameter
- Curvature
- Holes!



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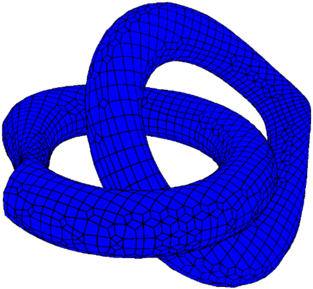

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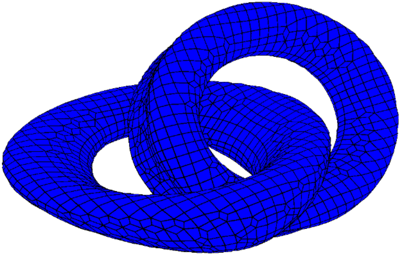

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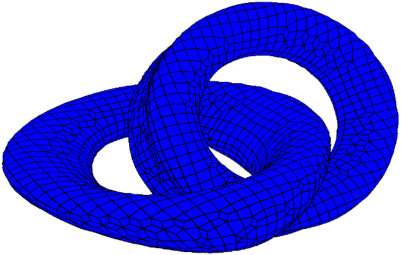

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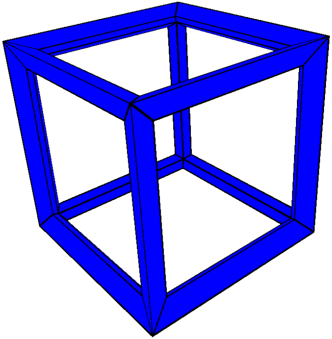

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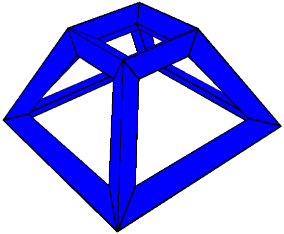
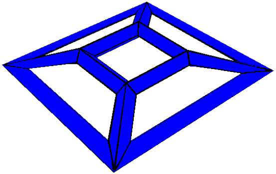


Number of holes: 5



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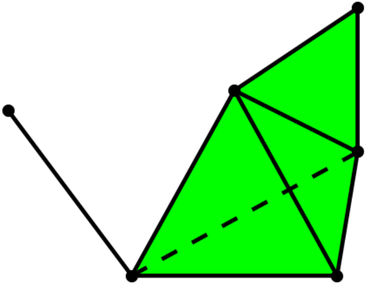
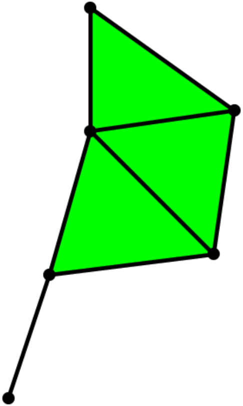
Simplicial complex

Union[1] of points, edges, triangles, tetrahedra, ... (simplices)





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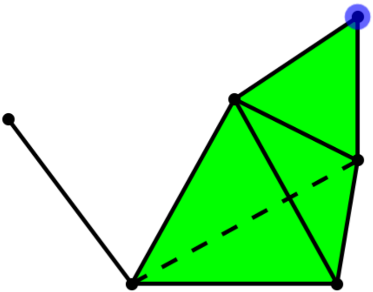

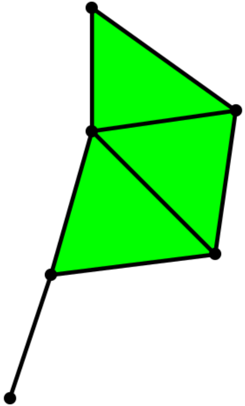

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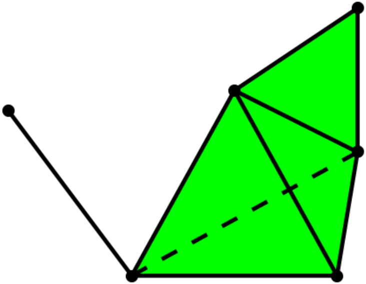

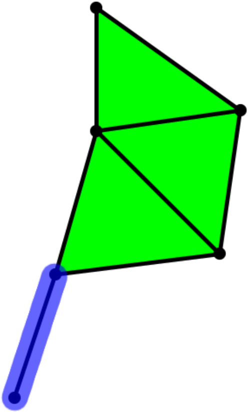

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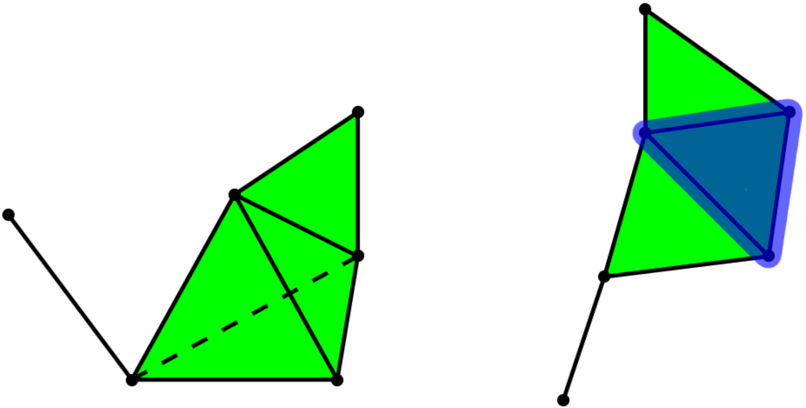

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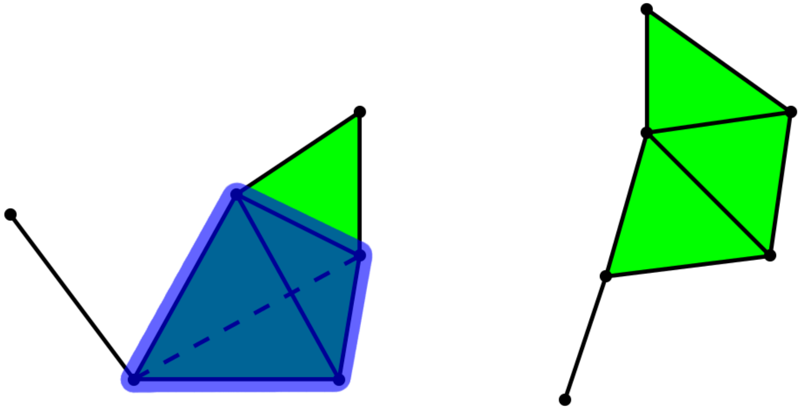

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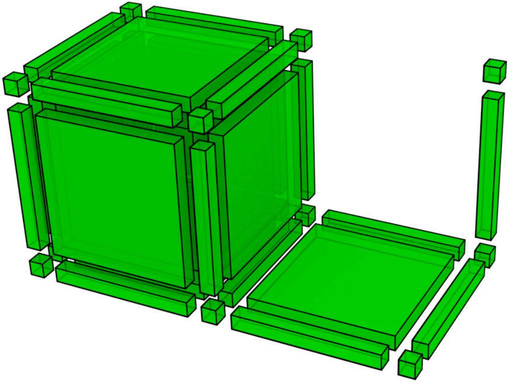
Union[2] of points, edges, squares, cubes, ... (cubes)



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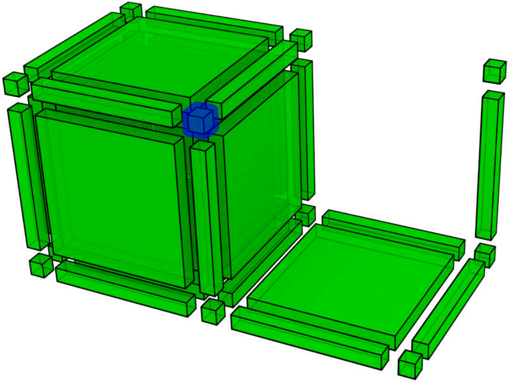

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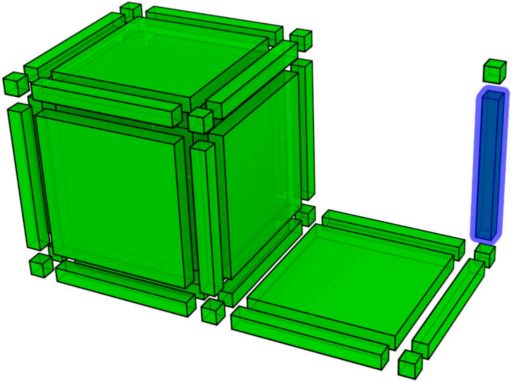

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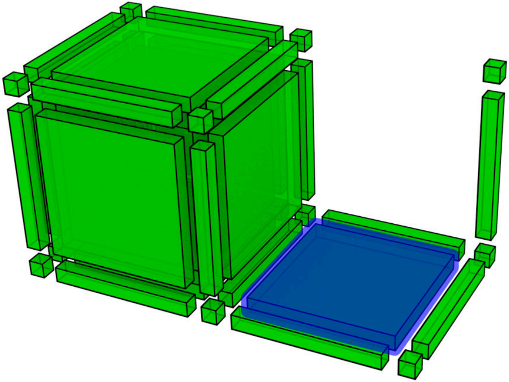

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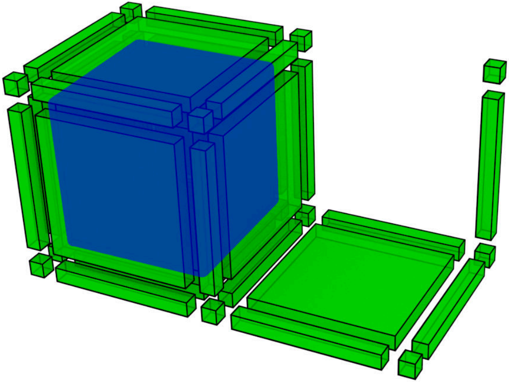

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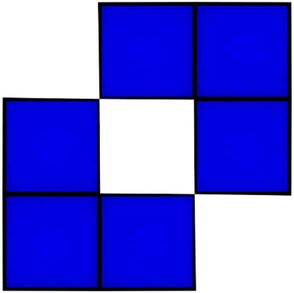
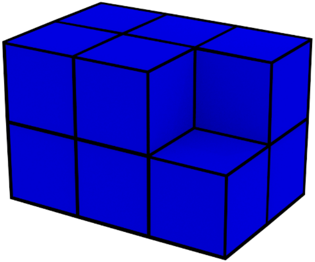
Discrete object

A nD discrete object is a subset of $\mathbb{Z}^n$





We usually choose a connectivity relation such as the $2n$ or the $(3^n - 1)$-connectivity.

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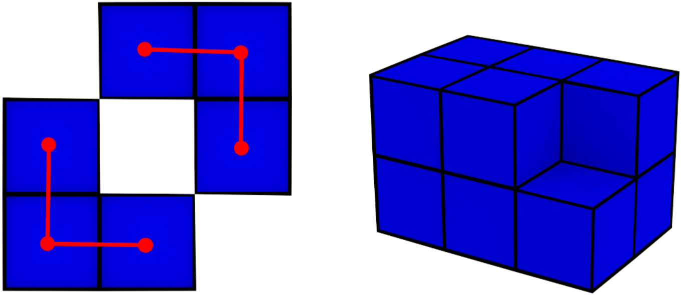

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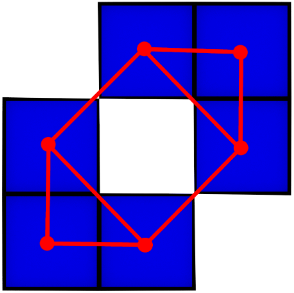
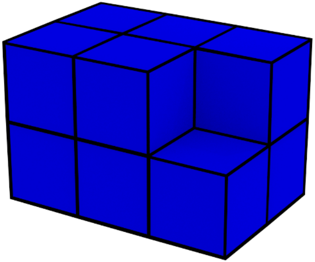
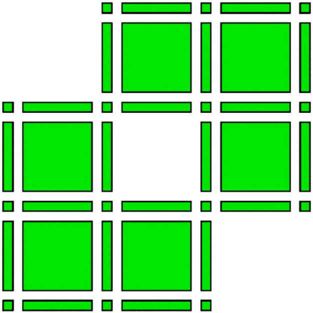
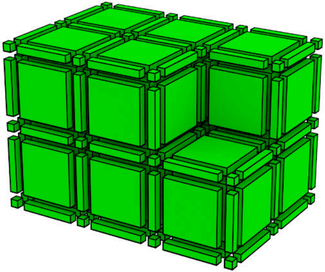




We usually choose a connectivity relation such as the $2n$ or the $(3^n - 1)$-connectivity.

We can transform a discrete object into a cubical complex in two ways, one for each connectivity relation.





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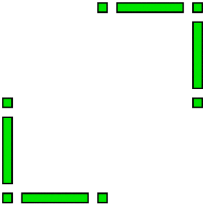

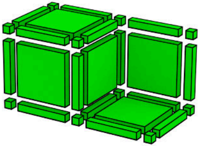

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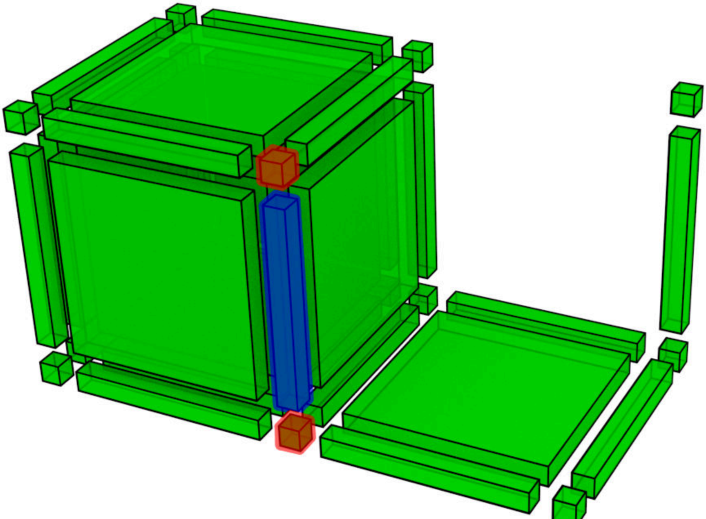
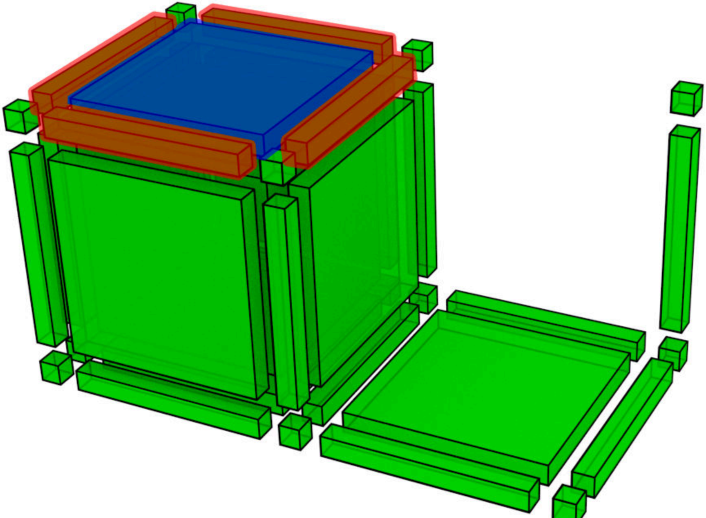
Red: its boundary (faces)



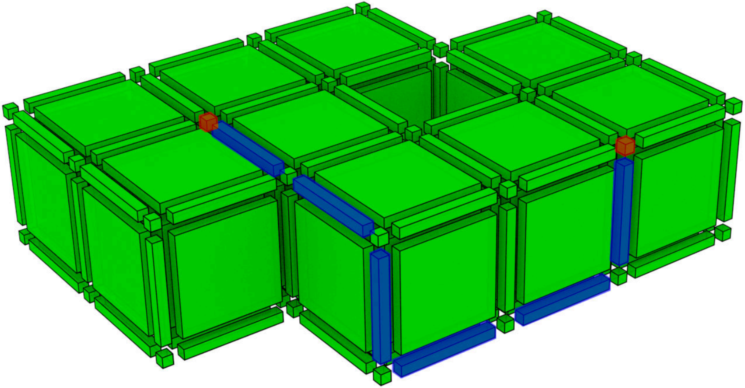
Blue: 2-cube

Red: its boundary (faces)



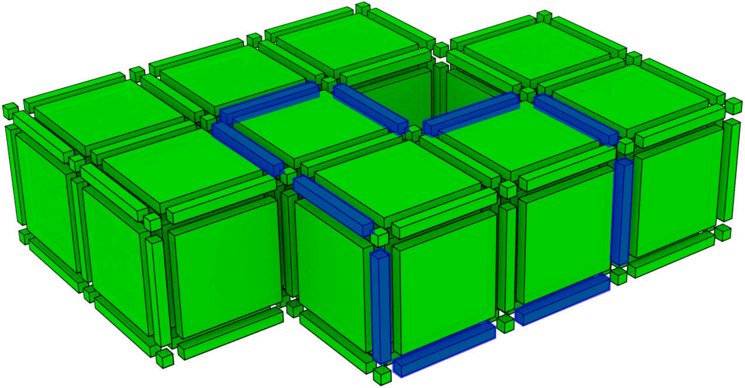
Blue: 1-chain

Red: its boundary



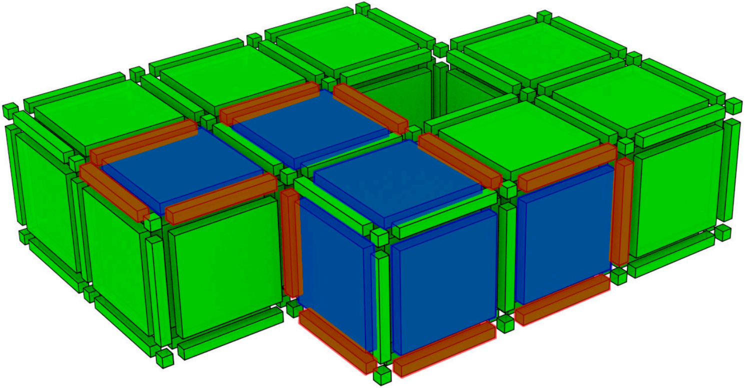
Blue: 1-chain (1-cycle)

Red: its boundary ($= \emptyset$)



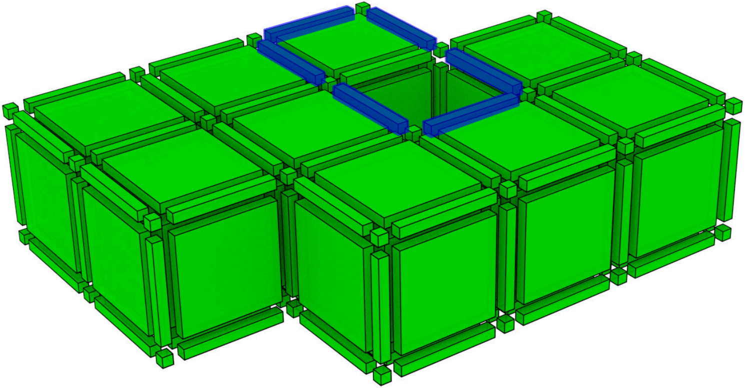
Blue: 2-chain

Red: its boundary (1-cycle)



Blue: 1-chain (1-cycle, but not boundary)

Red: its boundary ($= \emptyset$)



- K cubical complex, $\mathfrak{R}$ ring (e.g., $\mathbb{Z}$, $\mathbb{Z}_2$)
- Chain complex of K

$$\cdots C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

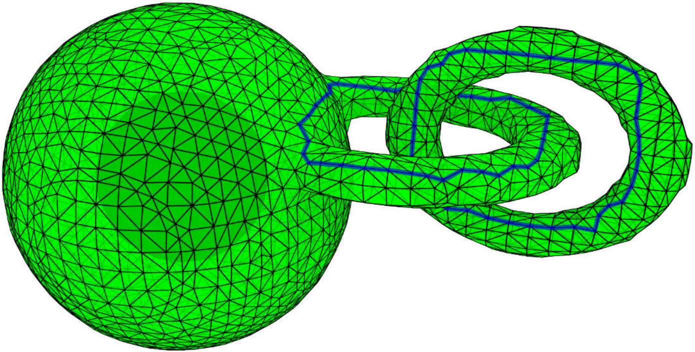
where $d_q d_{q+1} = 0 \Rightarrow \operatorname{im}(d_{q+1}) \subset \ker(d_q)$

- q-dimensional homology group
$H_q(K) := \ker(d_q)/\operatorname{im}(d_{q+1}) = \mathbb{Z}^{\beta_q} \oplus \mathbb{T}$
- q-dimensional Betti number: β_q

- $\beta_0 = \#$ connected components (0-holes)
- $\beta_1 = \#$ tunnels or handles (1-holes)
- $\beta_2 = \#$ cavities (2-holes)

Betti numbers are

- Topological invariants $\rightarrow$ classification
- Shape descriptors $\rightarrow$ understanding



$\beta_0 = 2,\ \beta_1 = 2,\ \beta_2 = 1,\ \beta_3 = 0,\ \ldots$

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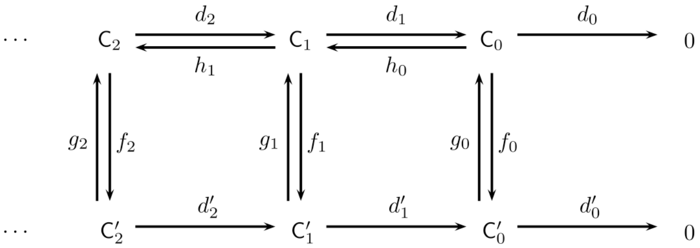
Effective homology theory [Sergeraert, 1992]

Reduction

Triplet $\rho = (h, f, g)$ of graded homomorphisms[3] between two chain complexes (C, d) and (C', d')



Both chain complexes have isomorphic homology groups

[3] with some conditions (cf. Definition 2.18)

A reduction is perfect if $d' = 0$. Hence

- $C' \cong H(C)$
- $g(C') =$ homology generators
- $f^*(C') =$ cohomology generators
- $d(x) = 0 \Rightarrow d(y) = x$ for $y = h(x)$

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2 The Homological Discrete Vector Field
- Introduction
- Definitions and theorems
- Computing a HDVF
- Deforming a HDVF
- Relation with other methods in computational homology
- Conclusion

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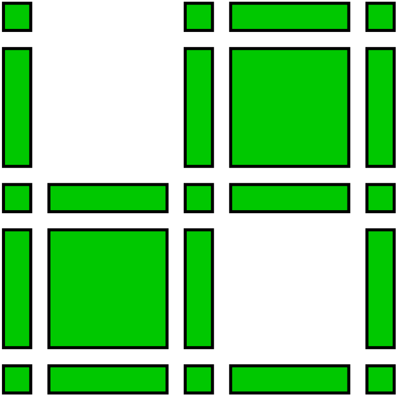
Discrete Morse theory [Forman, 1998]

- (CW-) complex
- Connectivity graph
- Matching $\mathcal{V}$
- Morse graph (no cycles)
- $\mathcal{V}$ is a discrete gradient vector field (DGVF)



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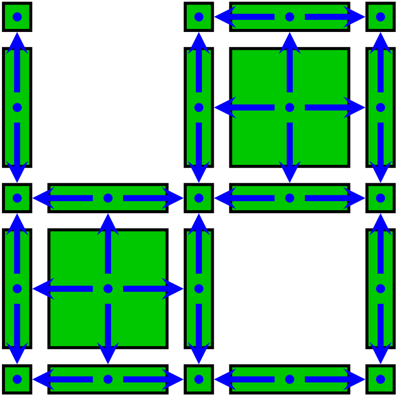

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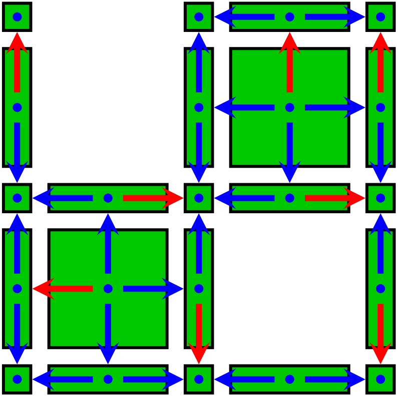

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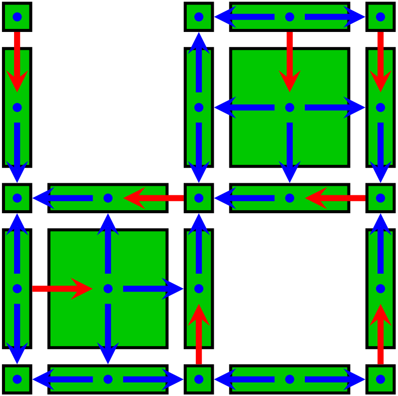

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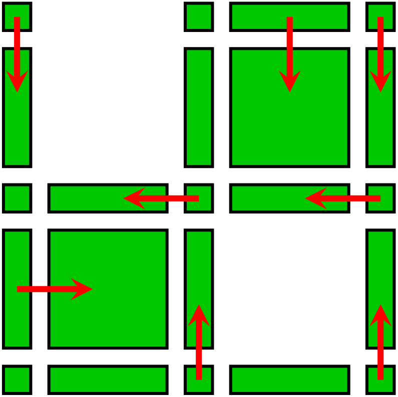
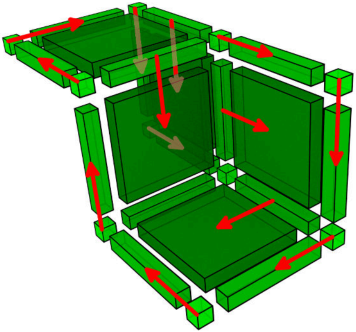
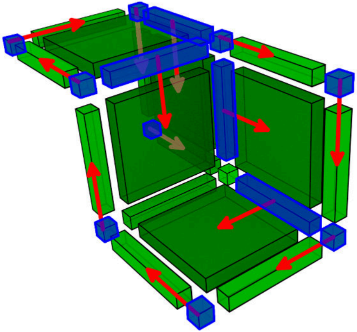




A DGVF

- Acyclic matching
- The arrows can be deduced from P and S
- It induces a reduction
- $|C_q| \geq \beta_q$



P: primary cells

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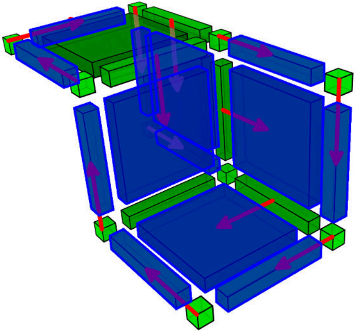

S: secondary cells

A DGVF

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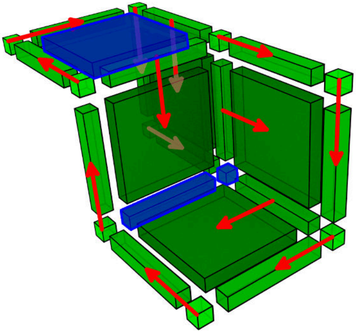

C: critical cells

A DGVF

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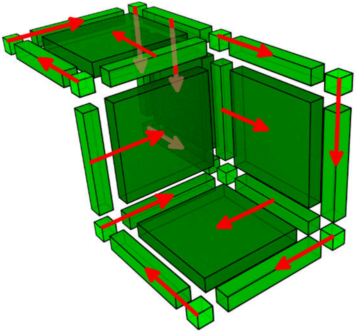


A perfect DGVF

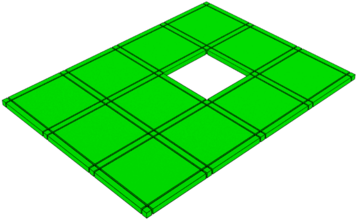
- Acyclic matching
- The arrows can be deduced from P and S
- It induces a perfect reduction
- $|C_q| = \beta_q$

So

- Algebra → graph theory
- Homology computation → optimization problem

But

- Finding optimal DGVF is NP
- No possible perfect DGVF always



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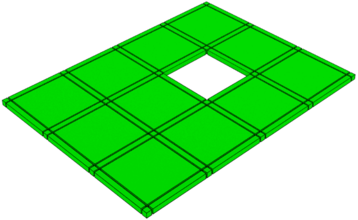

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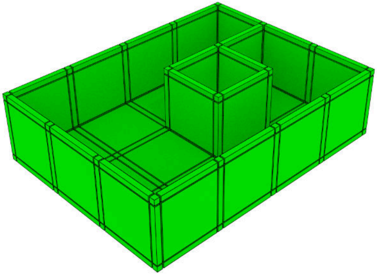

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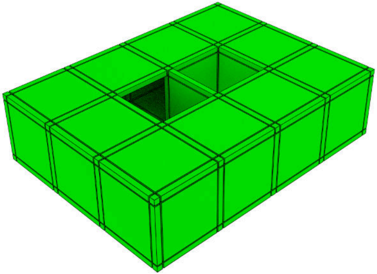

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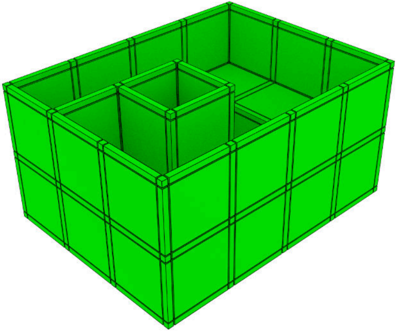

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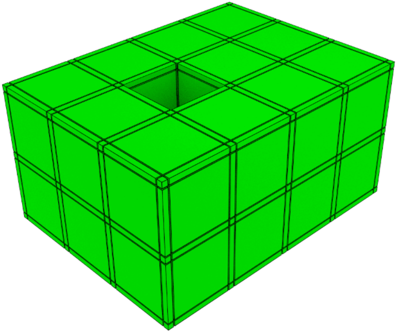
- Finding optimal DGVF is NP
- No possible perfect DGVF always



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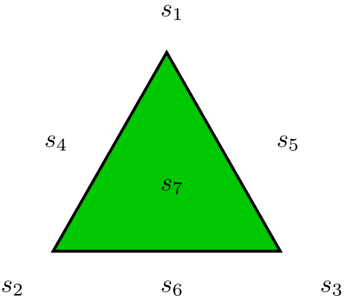
Boundary matrix

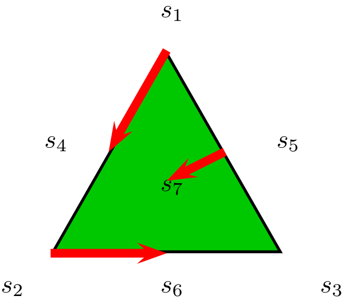
Matrix of the (linear) boundary operator d



$$d = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \\ s_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Boundary matrix

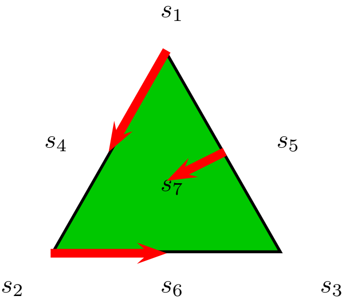
Matrix of the (linear) boundary operator *d*



$$d = \begin{array}{c|ccccccc} & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 \\ \hline s_1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ s_2 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ s_3 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ s_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ s_5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ s_6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ s_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Boundary matrix

Matrix of the (linear) boundary operator *d*



$$d(S)_{|P} = \begin{array}{c} \\ s_1 \\ s_2 \\ s_5 \end{array} \begin{array}{c} \begin{array}{ccc} s_4 & s_6 & s_7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

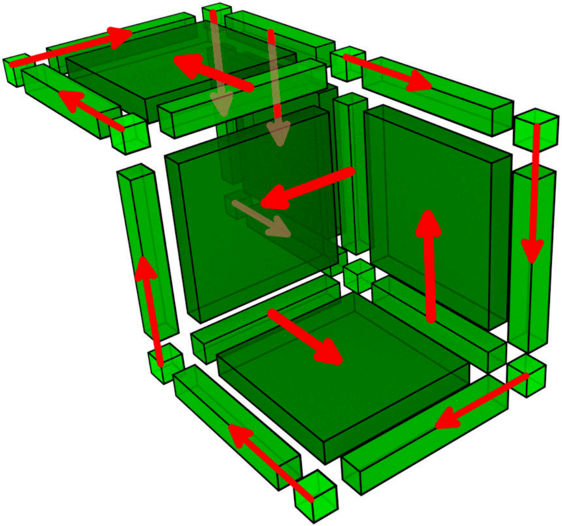
"Forget the cycles, focus on the reduction"

HDVF (Definition 3.1)

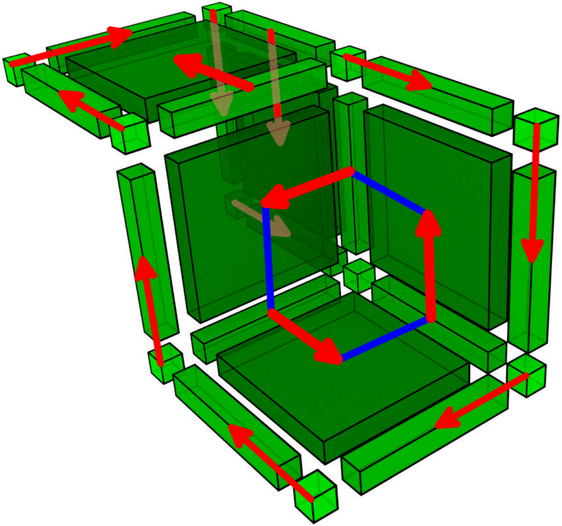
A homological discrete vector field (HDVF) $X = (P, S)$ on a CW complex K is a partition $K = P \sqcup S \sqcup C$ such that $d(S)_{|P}$ is an invertible matrix (in $\mathfrak{R}$)

We can always represent a HDVF as a discrete vector field (cf. Proposition 3.8)

Example: a HDVF with two cycles in the Morse graph



Example: a HDVF with two cycles in the Morse graph



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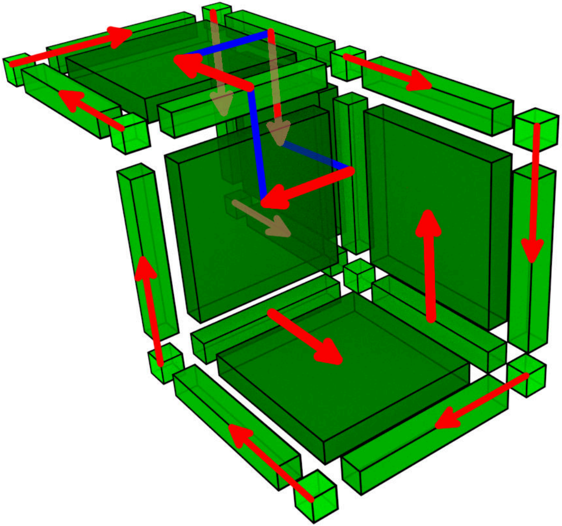
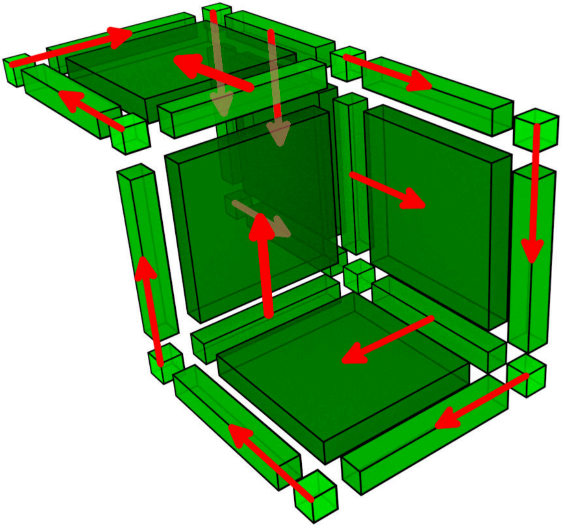
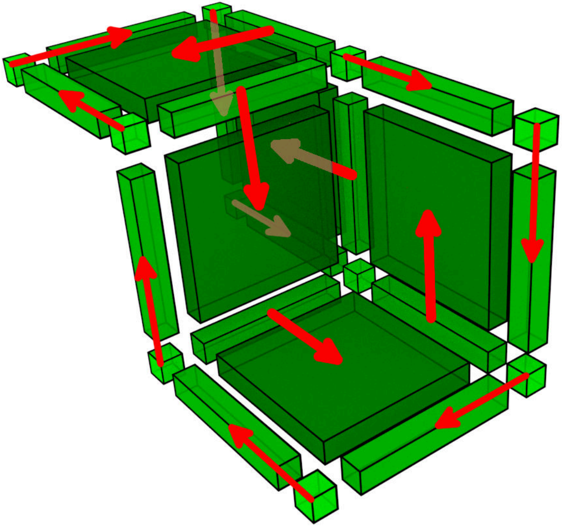


Example: a HDVF with two cycles in the Morse graph



Example: a HDVF with two cycles in the Morse graph



Theorem 3.9

Let K be a CW complex endowed with a HDVF X. Then X induces the reduction

$$(h, f, g) : (C, d) \Rightarrow (\mathfrak{R}[C], d')$$

where the operators h, f, g and the reduced boundary d' are given by

$$h = \begin{array}{c|ccc} & P & S & C \\ \hline P & 0 & 0 & 0 \\ S & H & 0 & 0 \\ C & 0 & 0 & 0 \end{array} \quad f = \begin{array}{c|ccc} & P & S & C \\ \hline C & F & 0 & I \end{array} \quad g = \begin{array}{c|c} & C \\ \hline P & 0 \\ S & G \\ C & I \end{array} \quad d' = \begin{array}{c|c} & C \\ \hline C & D \end{array}$$

$$H = (d(S)_{|P})^{-1}$$
$$F = -d(S)_{|C} \cdot H$$
$$G = -H \cdot d(C)_{|P}$$
$$D = d(C)_{|C} + F \cdot d(C)_{|P}$$

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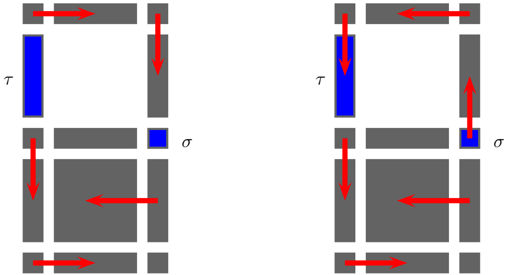
Proposition 3.12

K CW complex, $X = (P, S)$ HDVF, σ, τ critical cells. If $\langle d'(\tau), \sigma \rangle$ is a unit then $X' = (P \cup \{\sigma\}, S \cup \{\tau\})$ is a HDVF.



Algorithm 1: Compute a HDVF

Input: A CW complex K

Output: A HDVF X

```
repeat
    Find two critical cells σ, τ such that ⟨d′(τ), σ⟩ is a unit;
    Add (σ, τ) to X;
    Update the reduced boundary matrix D;
until idempotency;
```

Theorem 3.15

Algorithm 1 can be computed within $\mathcal{O}(n^3)$ operations.

Algorithm 2: Compute a HDVF

Input: A CW complex K

Output: A HDVF X

```
repeat
    Find two critical cells σ, τ such that ⟨d'(τ), σ⟩ is a unit;
    Add (σ, τ) to X;
    Update the reduced boundary matrix D;
until idempotency;
```

Theorem 3.15

Algorithm 1 can be computed within $\mathcal{O}(n^3)$ operations.

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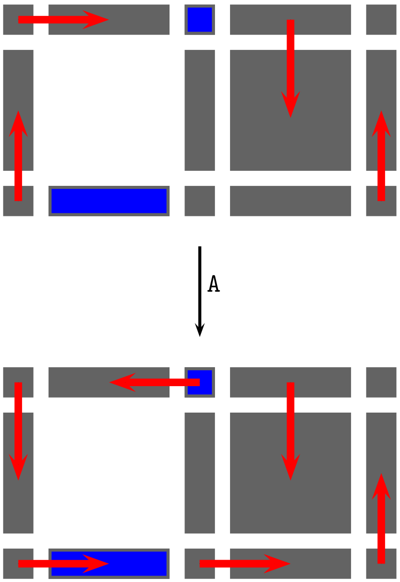
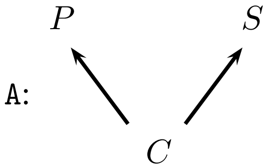
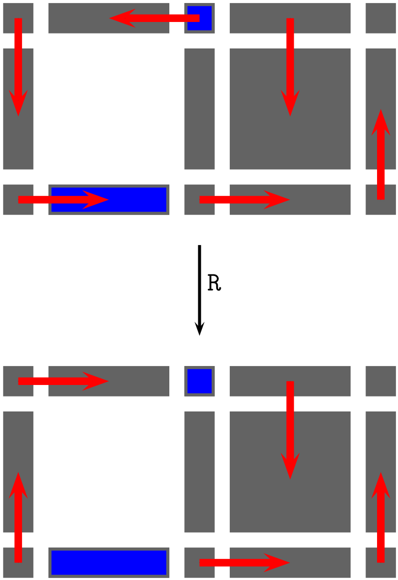
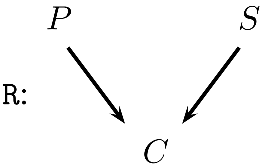
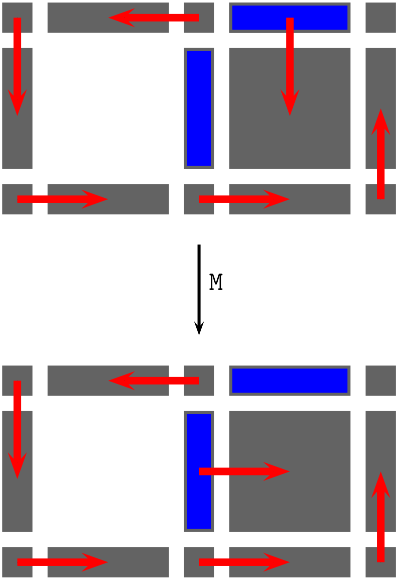
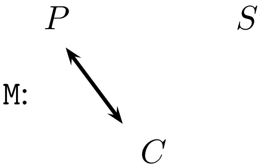

A


A:
P
S
C


R


P
S
R:
C


M


P
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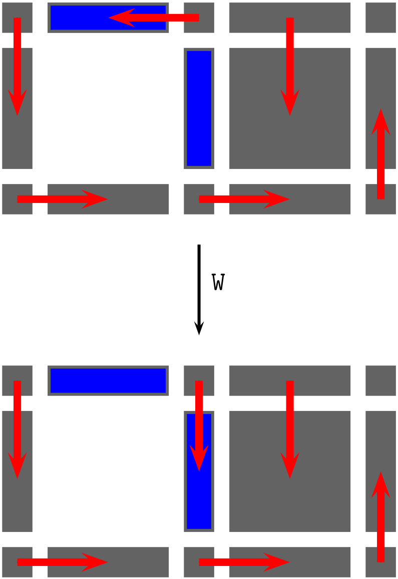
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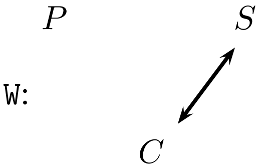
W:
P
S
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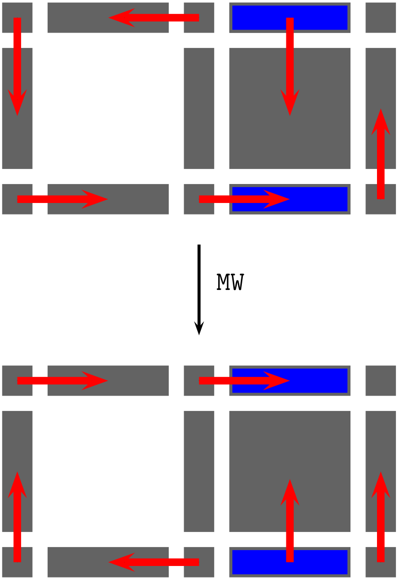

$$P \longleftrightarrow S$$

MW:

$$C$$

Proposition 3.19

Let K be a CW complex endowed with a HDVF X. Let $\sigma \in P$, $\tau \in S$ and $\gamma, \gamma' \in C$. Thus,

1. $\mathtt{A}(X, \gamma, \gamma')$ is a HDVF if $\langle d'(\gamma'), \gamma \rangle$ is a unit
2. $\mathtt{R}(X, \sigma, \tau)$ is a HDVF if $\langle h(\sigma), \tau \rangle$ is a unit
3. $\mathtt{M}(X, \sigma, \gamma)$ is a HDVF if $\langle f(\sigma), \gamma \rangle$ is a unit
4. $\mathtt{W}(X, \tau, \gamma)$ is a HDVF if $\langle g(\gamma), \tau \rangle$ is a unit
5. $\mathtt{MW}(X, \sigma, \tau)$ is a HDVF if $\langle dh(\sigma), \tau \rangle$ and $\langle hd(\tau), \sigma \rangle$ are units

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Proposition 3.21

Every DGVF is a HDVF.

Proposition 3.22

Every iterated DGVF[4] is a HDVF.

Proposition 3.23

Let K be a CW complex. Then,

1. Algorithm 1 performs a partial diagonalization of the boundary matrices of K;
2. Algorithm 1 computes a perfect HDVF whenever $\mathfrak{R}$ is a field.

Thus, we can compute persistent homology with the HDVF

[4] [Dlotko and Wagner, 2012]

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Conclusion:

- HDVF: combinatorial structure for computing homology
- Visually representable
- More powerful than DGVF
- Cubical complexity

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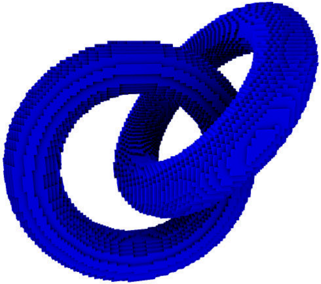
We want to compute the Betti numbers of binary volumes



It seems that:

- $\beta_0 = \#$ connected components
- $\beta_2 = \#$ bounded connected components of the complement
- β_1 ?

[Delfinado and Edelsbrunner, 1995], [Dey and Guha, 1998]: 3D simplicial complexes

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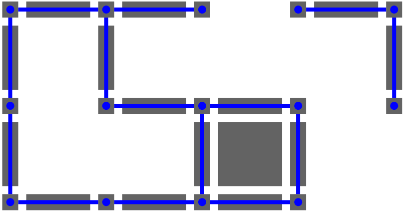
Ingredients

(1) β_0 is the number of connected components
(2) Duality
(3) Euler-Poincaré formula

Ingredient (1)

Let K be a 3D cubical complex. Consider the graph $G_0(K)$



Proposition 4.3

$\beta_0(K) =$ number of connected components in $G_0(K)$

Ingredient (2)

Proposition 4.4

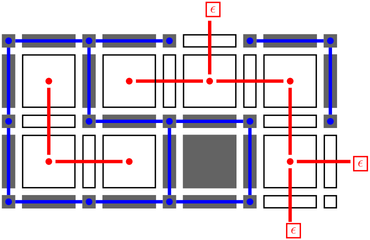
Let $K \subset L$ be two 3D cubical complexes such that $\beta(L) = (1, 0, 0, 0)$. Then,

$$\beta_q(K) = \begin{cases} \beta_1(L - K) + 1 & \text{if } q = 0 \\ \beta_{q+1}(L - K) & \text{else} \end{cases}$$

Thus, $\beta_2(K) = \beta_3(L - K)$

Ingredient (2)

Let $K \subset L$ be two 3D cubical complexes. Consider the graph $G_3(L - K)$



Proposition 4.5

$\beta_3(L - K) =$ number of connected components in $G_3(L - K)$ minus one.

Ingredient (3)

Euler-Poincaré Formula

$$\begin{aligned}\chi(K) &= |K_0| - |K_1| + |K_2| - |K_3| \\ &= \beta_0(K) - \beta_1(K) + \beta_2(K)\end{aligned}$$

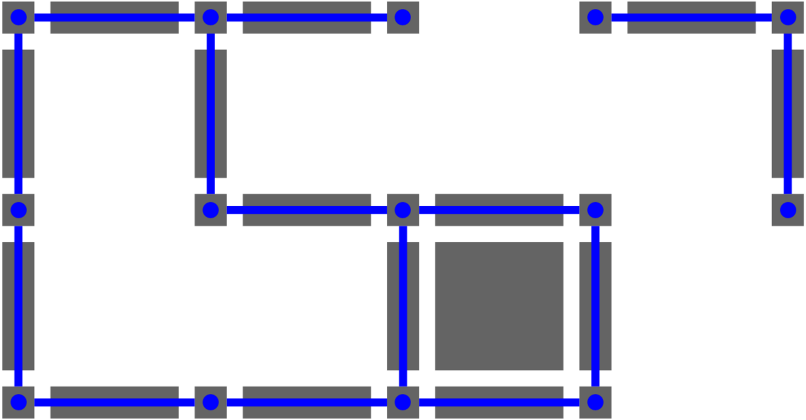
Thus, $\beta_1(K) = \beta_0(K) + \beta_2(K) - \chi(K)$

Computing the Betti numbers

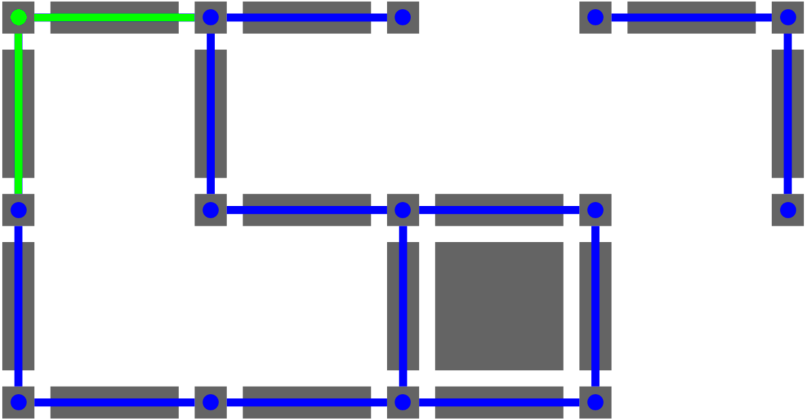
1. $\beta_0 \leftarrow$ number of connected components of $G_0(K)$
2. $\beta_2 \leftarrow$ number of connected components of $G_3(L - K)$ - 1
3. $\beta_1 \leftarrow \beta_0 + \beta_2 - \chi(K)$

- Linear time and space complexity
- We propose two versions for implementing this method

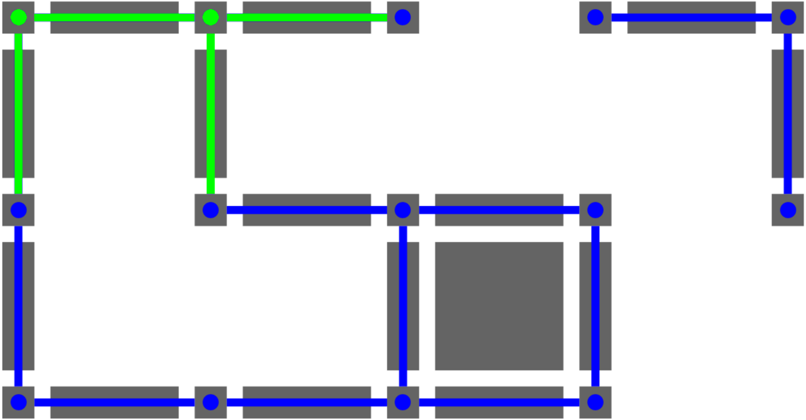
Sequential algorithm: BFS, iterative



Sequential algorithm: BFS, iterative



Sequential algorithm: BFS, iterative



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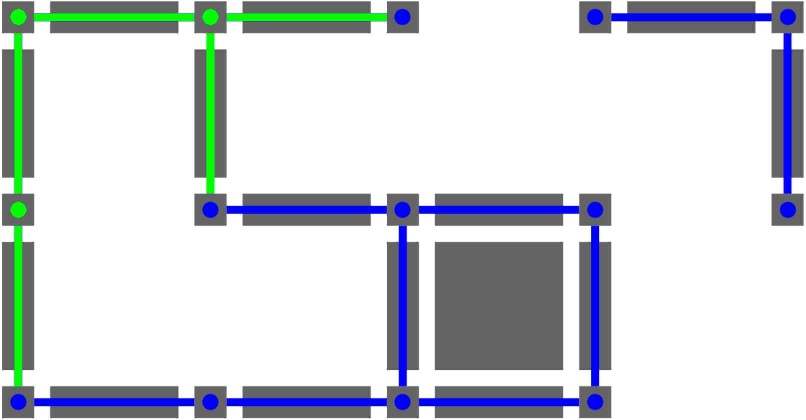
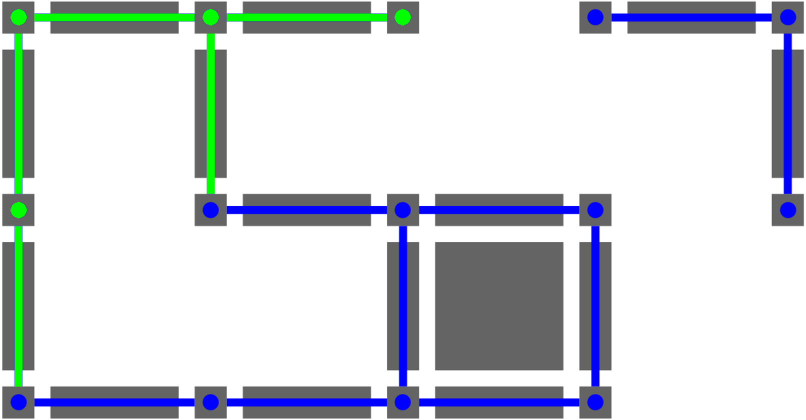


Sequential algorithm: BFS, iterative



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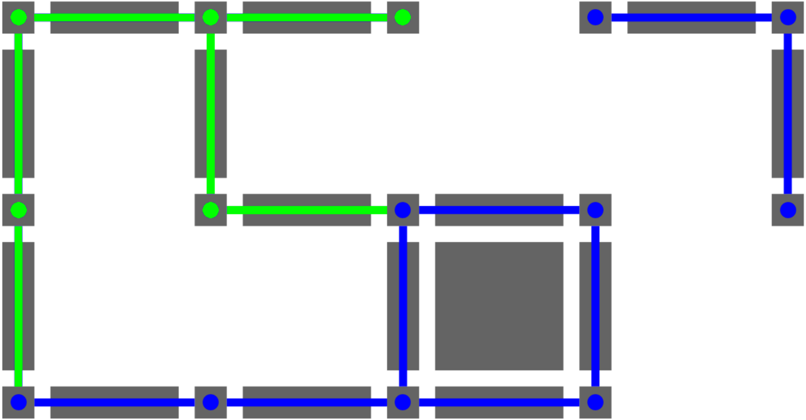

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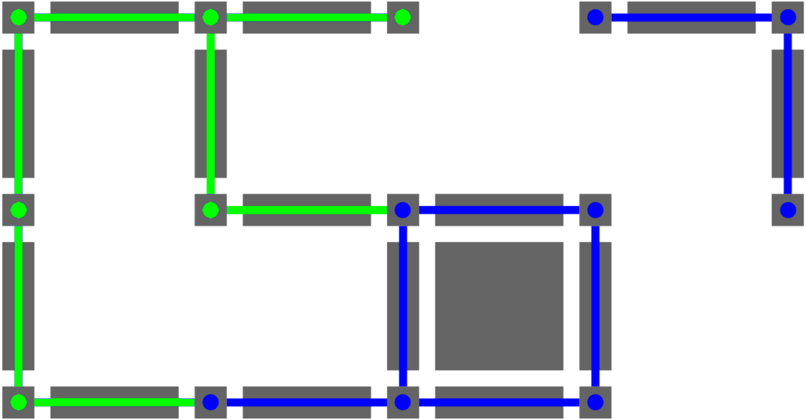

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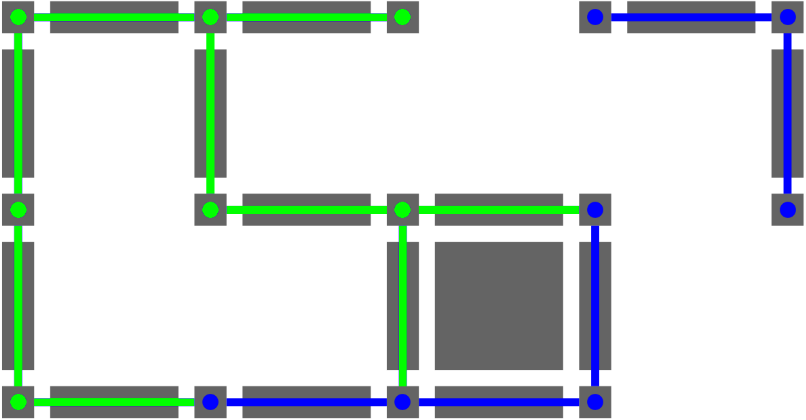

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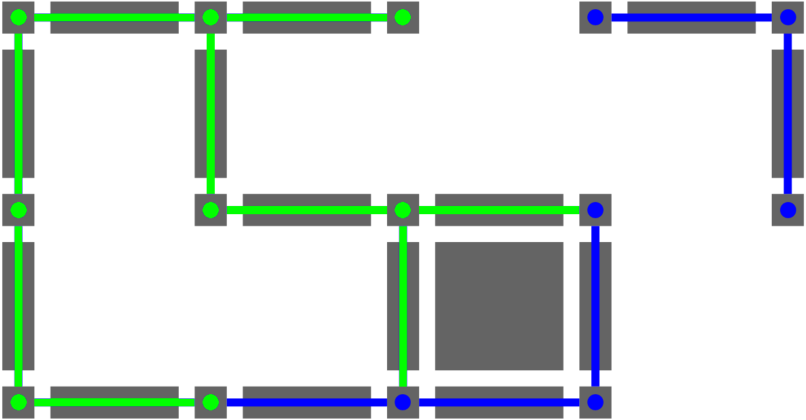

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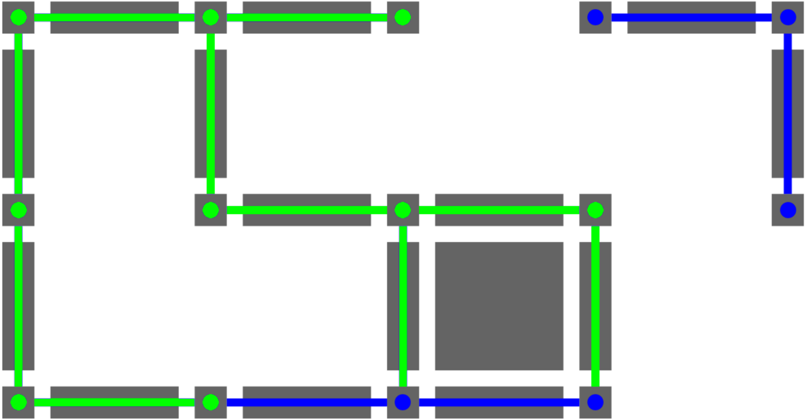

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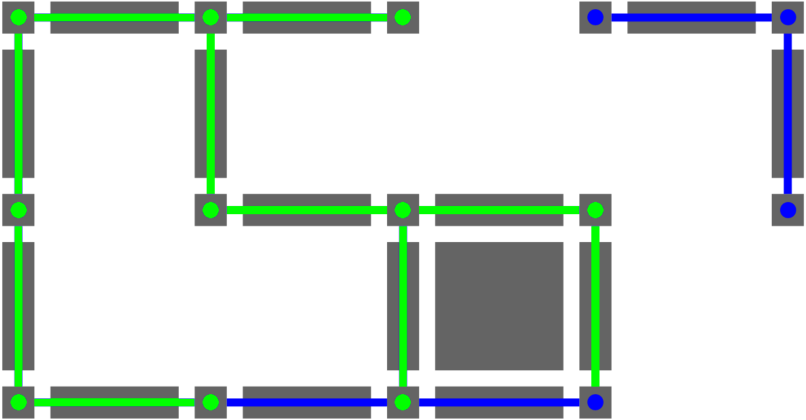

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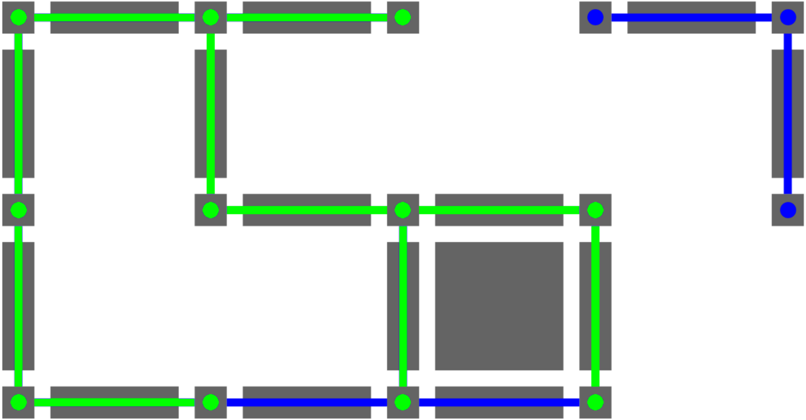

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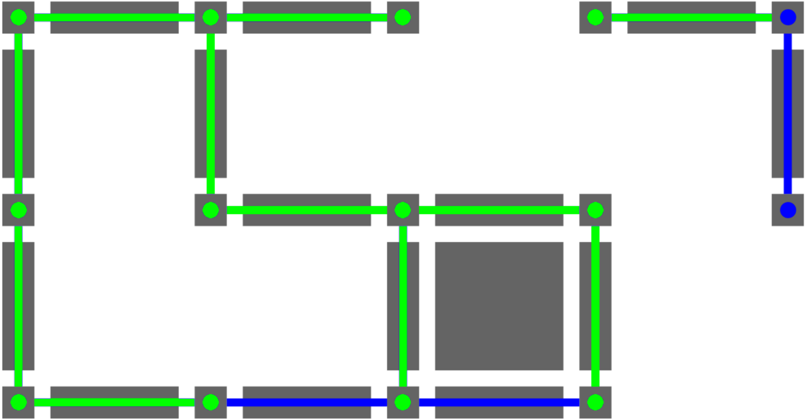

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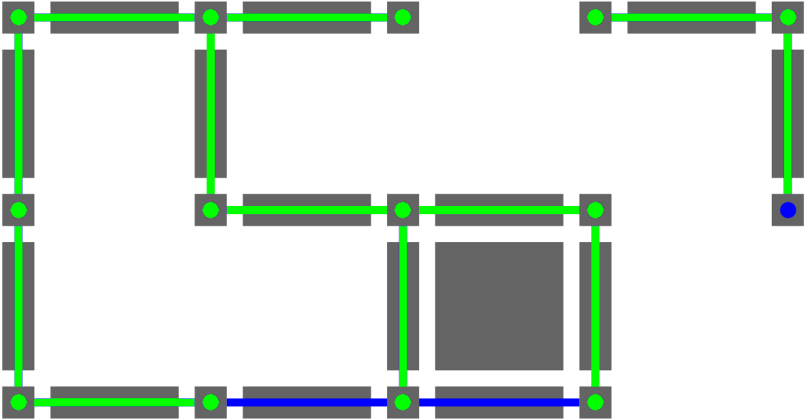
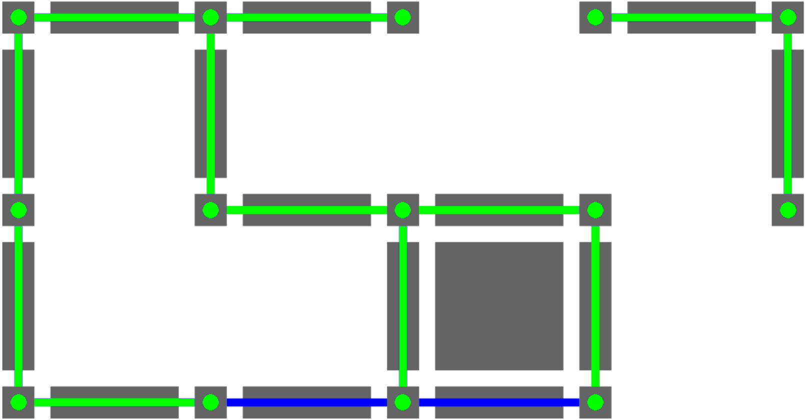
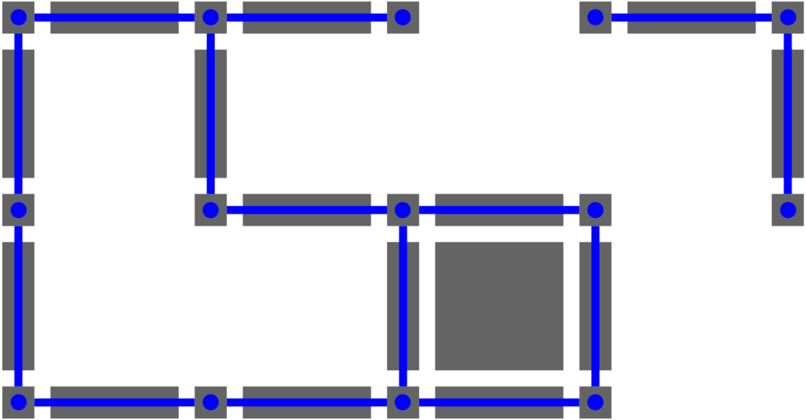


Sequential algorithm: BFS, iterative



Recursive algorithm: divide-and-conquer, union-find data set, partial parallelization



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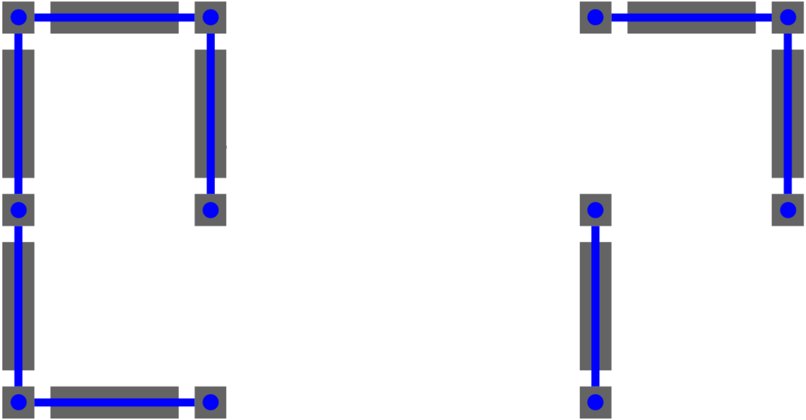

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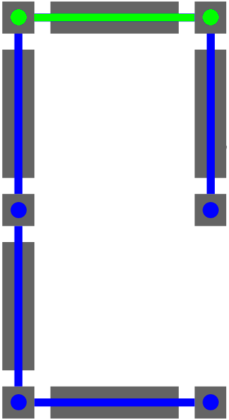

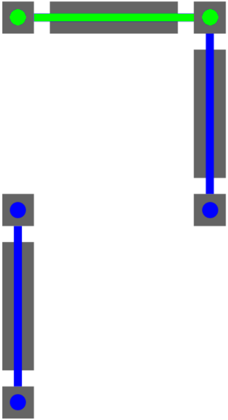

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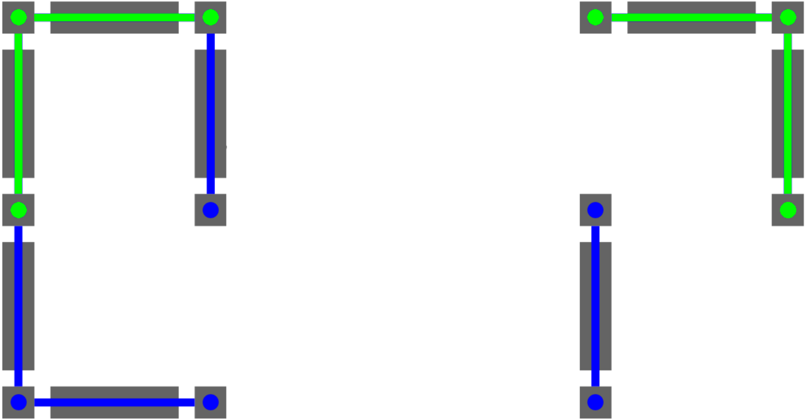

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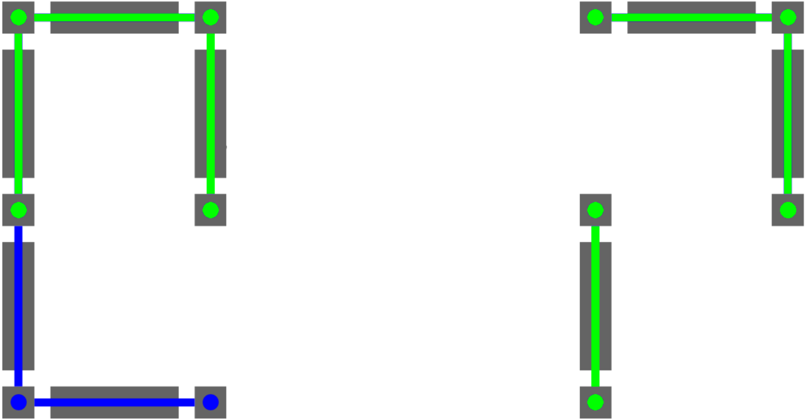
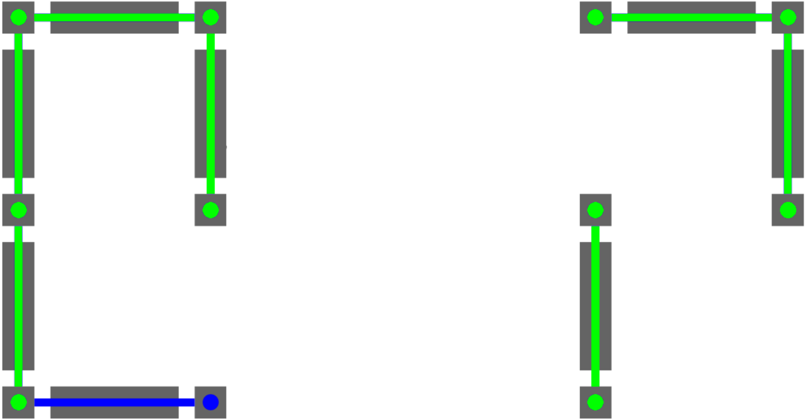


Recursive algorithm: divide-and-conquer, union-find data set, partial parallelization



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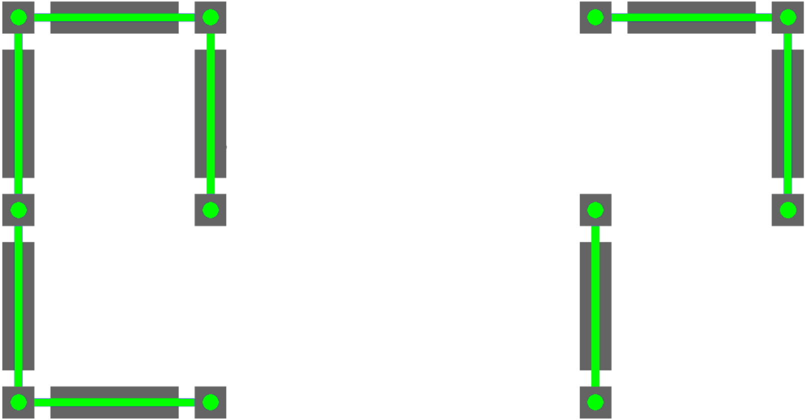

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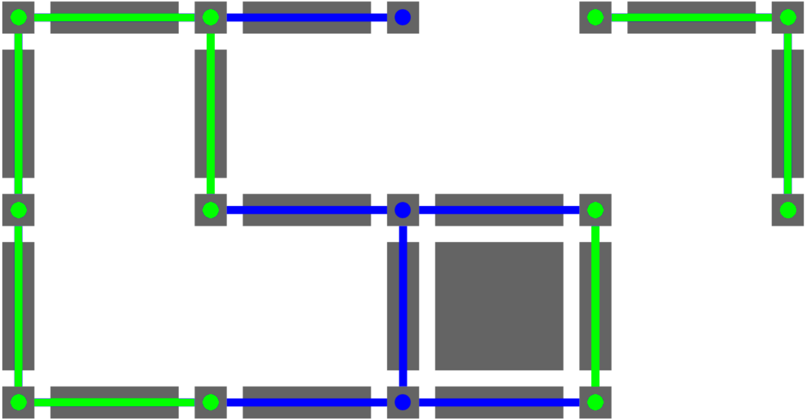

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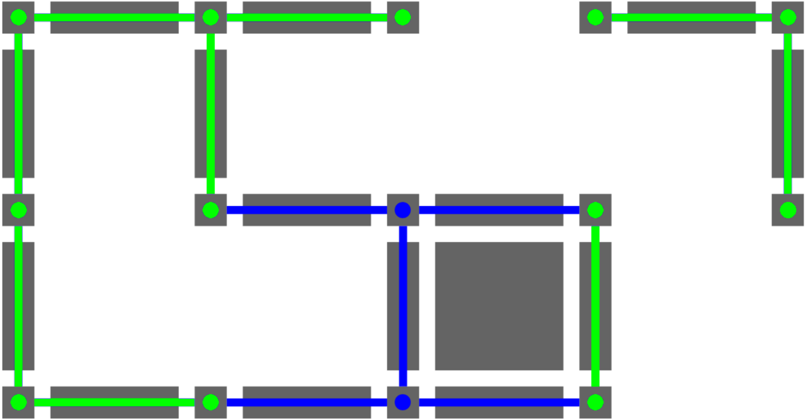

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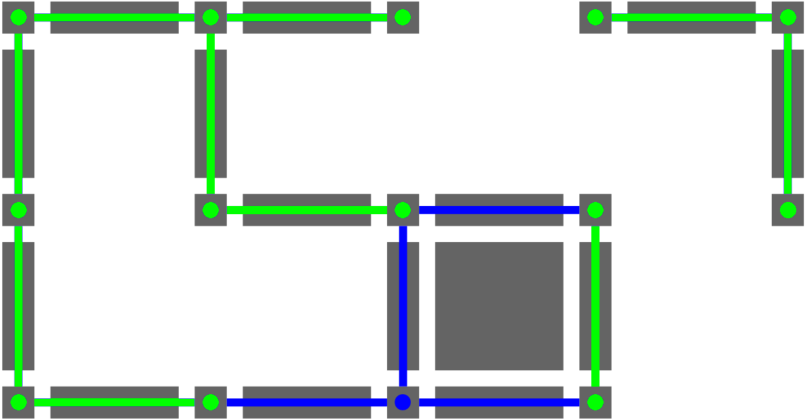

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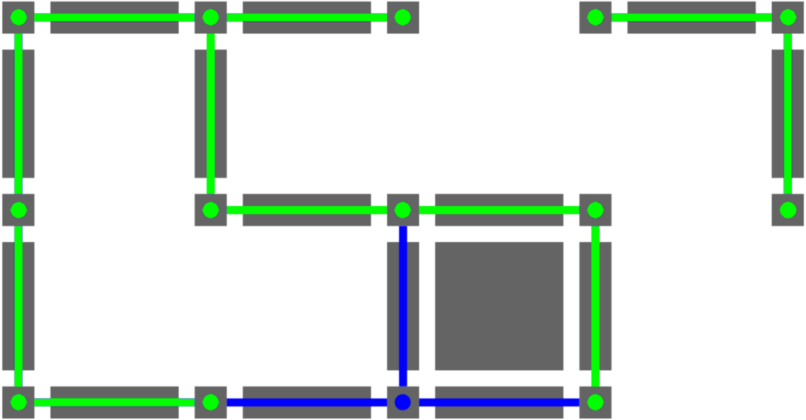

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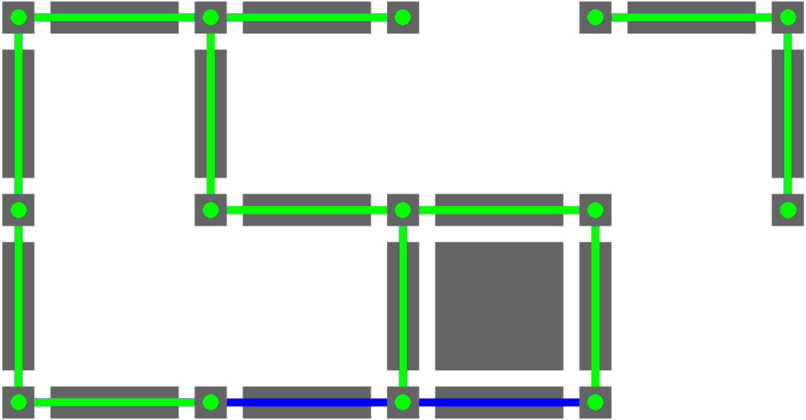

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Comparison against `CAPD::RedHom`[5] library

Size	RedHom	VB-s	VB-r	VB-rp
51^3	0.1842	0.0026	0.0026	0.0023
101^3	1.268	0.0142	0.0148	0.0091
201^3	10.78	0.1309	0.1232	0.0552
301^3	40.89	0.4303	0.4176	0.1583
401^3	101.26	1.436	0.983	0.3092
501^3	—	3.609	1.977	0.5494

Table: Execution time (in seconds) versus the size of the cubical complex.

Space is the problem, not time.

[5] [Juda and Mrozek, 2014]

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Conclusion:

- Simple algorithm for Betti numbers relying on connected components computation
- Linear time complexity
- Combinatorial and constructive proofs
- Implementation published under GNU GPL v3
- More effective than available algorithms

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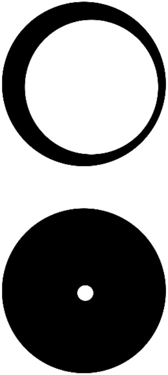
5 Conclusion

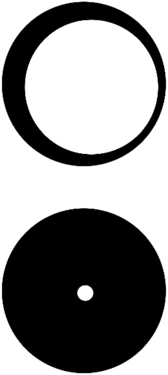
- We can know **how many** holes there are in an object
- We cannot know **where** or **how** they are

Size of a hole

The 1st one is *bigger* than the 2nd one



The 2nd one is *thicker* than the 1st one



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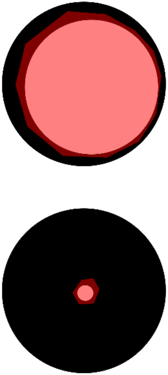

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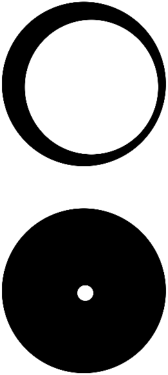

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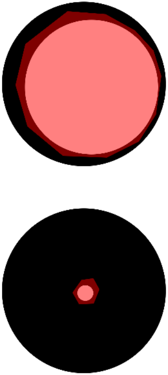

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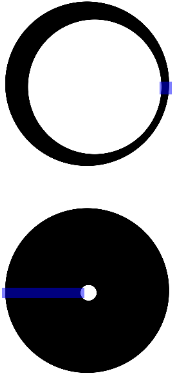
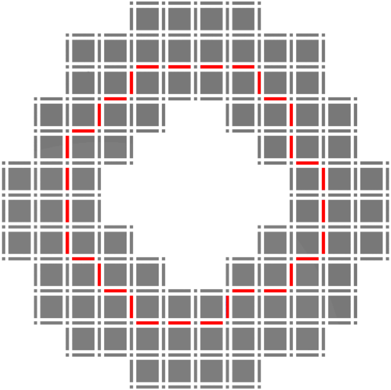
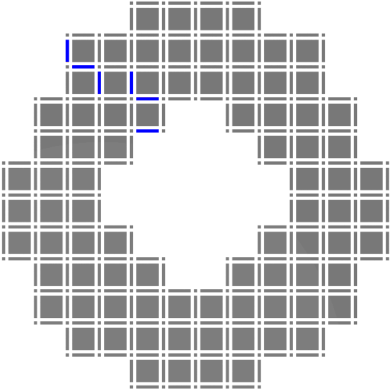


Representing a hole

Homology

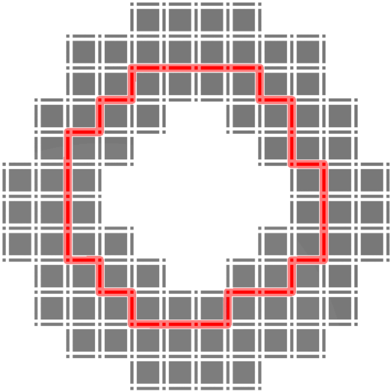


Cohomology

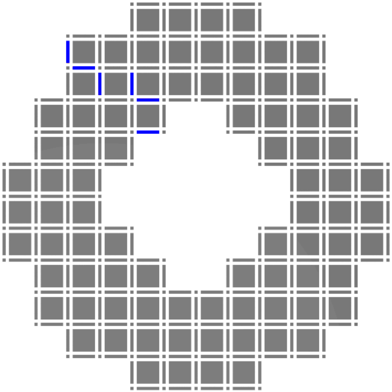


Representing a hole

Homology

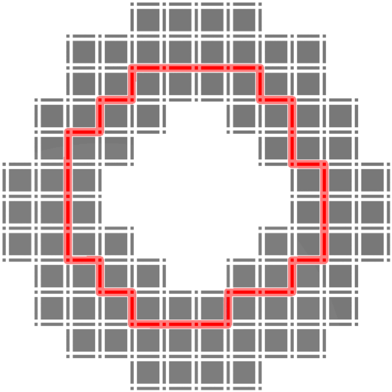


Cohomology



Representing a hole

Homology



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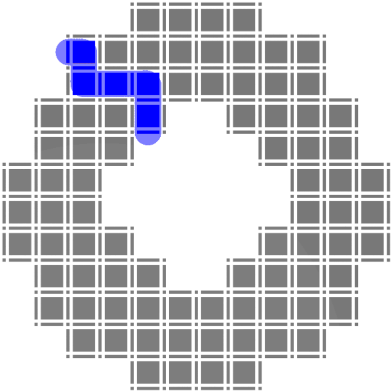
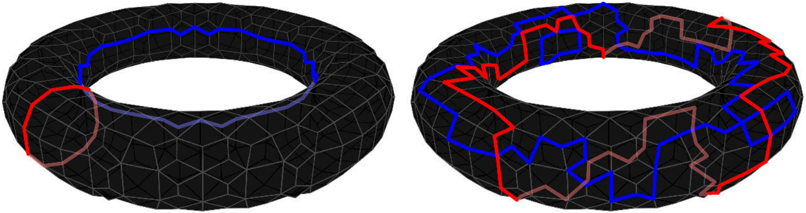


Representing a hole

Do homology generators really represent holes?



Geometry	+	Topology
↓		↓
Signed distance transform		Persistent homology

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Signed distance transform

Let O be a discrete object,

$$sdt_O(x) = \begin{cases} -d(x, O^c) & \text{if } x \in O \\ d(x, O) & \text{if } x \notin O \end{cases}$$

Figure: Sublevel sets of the signed distance form

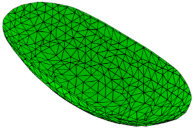
Persistent homology

Given a filtration F, we can define its set of persistence intervals through its persistent homology groups.

- These intervals tell the lifetime of the holes in the filtration
- They are represented as a set of points $PD(F)$ in $\mathbb{R}^2$

Example

- height: 5
- β_0: **1**
- β_1: 0
- β_2: 0

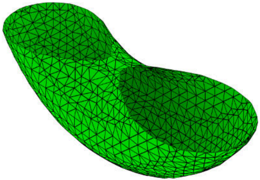


Persistence intervals:

- Dimension 0:
- Dimension 1:
- Dimension 2:

Example

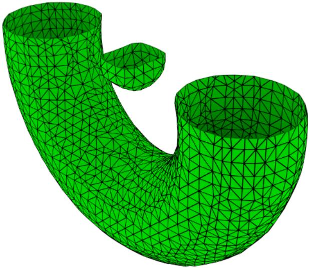
- height: 7
- β_0: 1
- β_1: **1**
- β_2: 0



Persistence intervals:

- Dimension 0:
- Dimension 1:
- Dimension 2:

Example

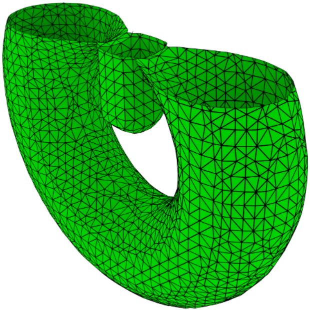


- height: 11
- β_0: **2**
- β_1: 1
- β_2: 0

Persistence intervals:

- Dimension 0:
- Dimension 1:
- Dimension 2:

Example

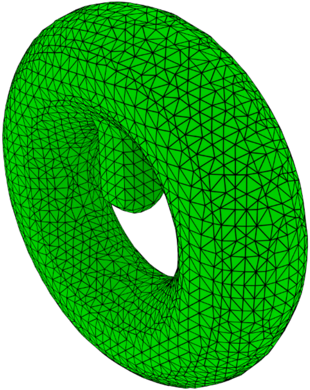


- height: 13
- β_0: **1**
- β_1: **2**
- β_2: 0

Persistence intervals:

- Dimension 0: $(10, 13)$
- Dimension 1:
- Dimension 2:

Example

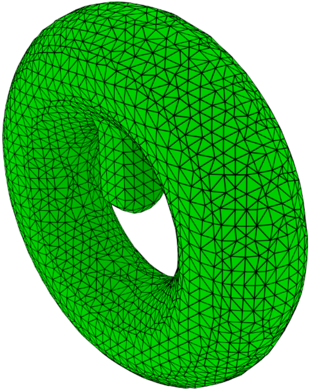


- height: 17
- β_0: 1
- β_1: 2
- β_2: **1**

Persistence intervals:

- Dimension 0: $(10, 13)$
- Dimension 1:
- Dimension 2:

Example

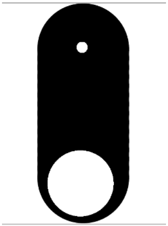


- height:
- β_0: 1
- β_1: 2
- β_2: 1

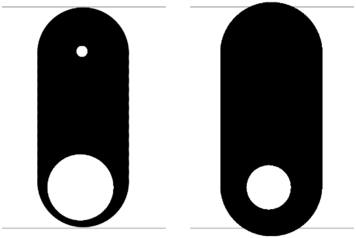
Persistence intervals:

- Dimension 0: $(10, 13)$, $(0, \infty)$
- Dimension 1: $(7, \infty)$, $(13, \infty)$
- Dimension 2: $(17, \infty)$

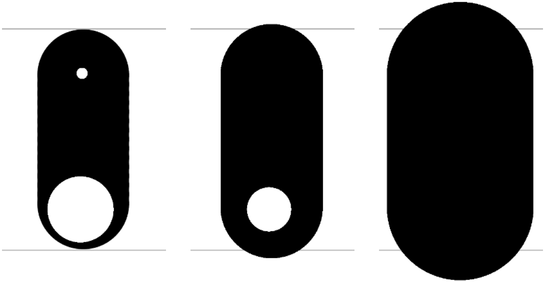
Persistent homology with signed distance transform



Persistent homology with signed distance transform



Persistent homology with signed distance transform



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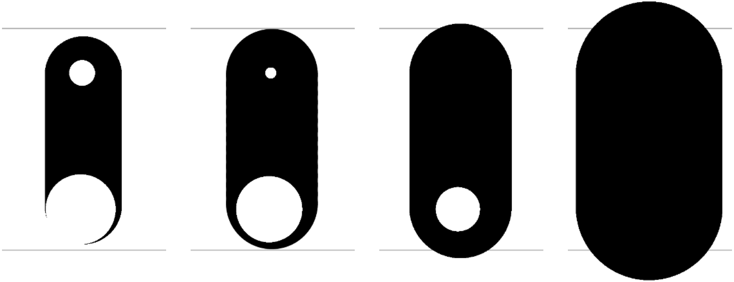

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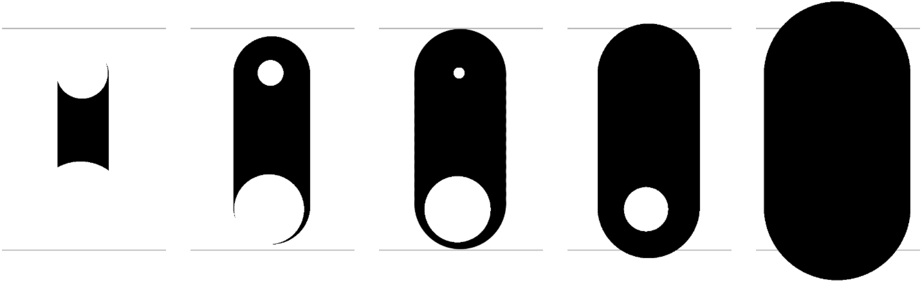

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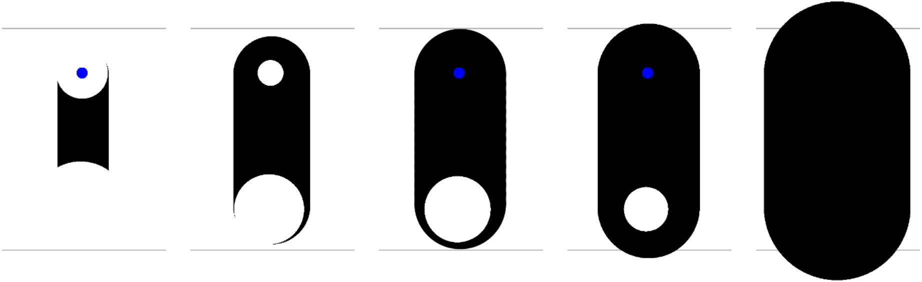
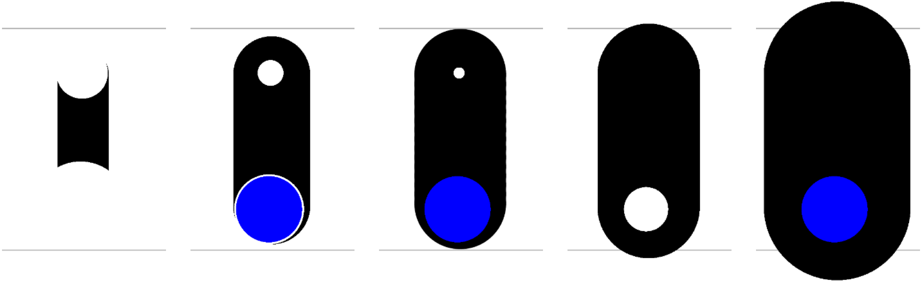


Persistent homology with signed distance transform



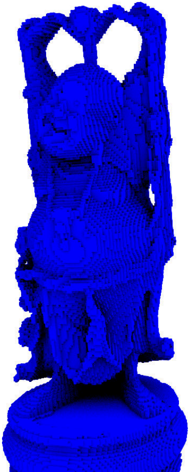
Thickness and breadth (Definition 5.1)

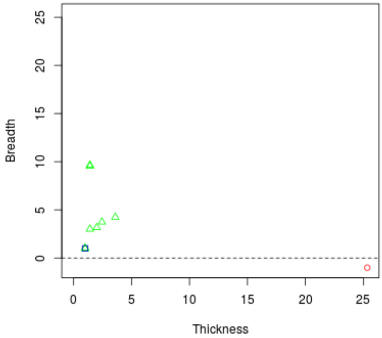
Let O be a discrete object and F the filtration defined by the sublevel sets of its signed distance transform. Let $TB(O) = \{(-x, y) \in PD(F) \mid x \leq 0, y \geq 0\}$. Its intervals are the thickness-breadth pairs of O

- There is a thickness-breadth pair (t, b) for each hole of O
- t is the thickness of the hole and b, its breadth

Thickness-breadth diagram

Thickness-breadth pairs can be represented like persistence diagrams





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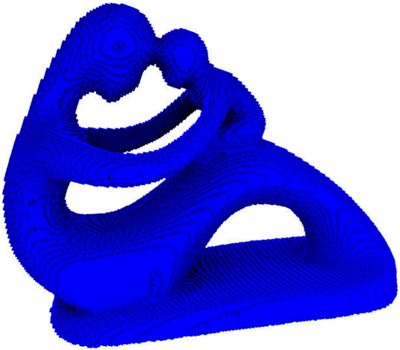

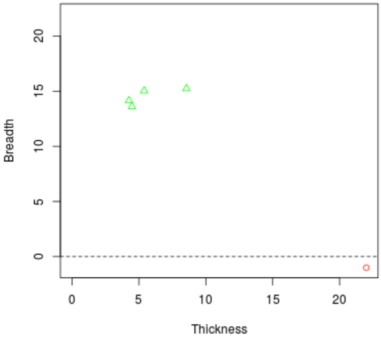

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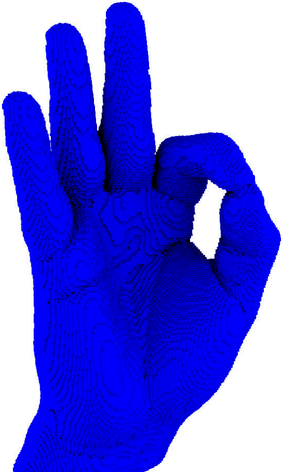

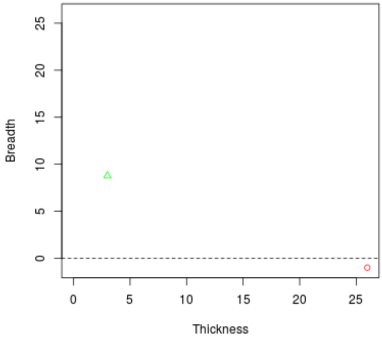

Theorem 5.2

Let X and Y be two 3D discrete objects. Let us call

$$\delta = d_H(X, Y) + d_H(\mathbb{Z}^3 \setminus X, \mathbb{Z}^3 \setminus Y) + 2\sqrt{3}$$

Thus, for every thickness-breadth pair $p_X = (x, y)$ of X such that $x, y > \delta$, there exists another thickness-breadth pair $p_Y = (x', y')$ of Y such that

$$||p_X - p_Y||_\infty \leq \delta$$

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Thickness and breadth ball

Let be (t, b) a TB-pair and (σ, τ) its pair of cells

- The thickness ball of (t, b) is the ball of radius t centered at σ
- The breadth ball of (t, b) is the ball of radius b centered at τ

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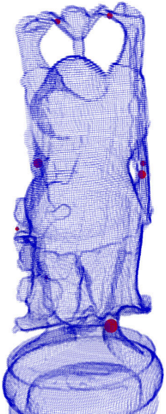

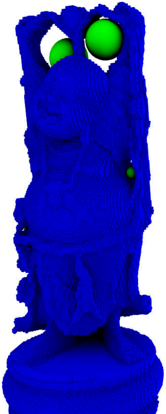

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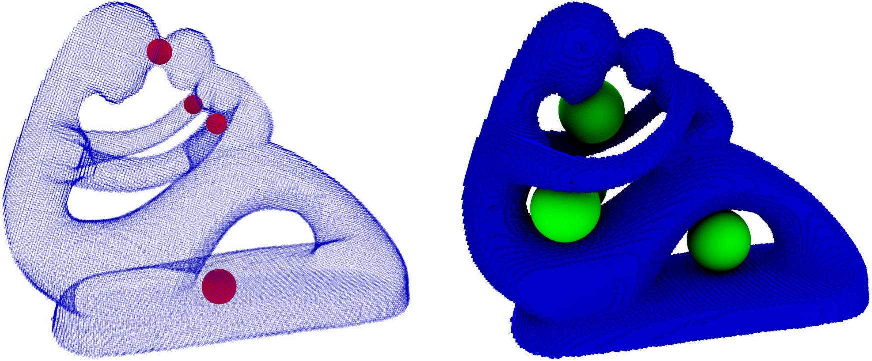

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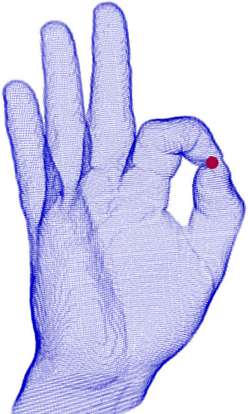
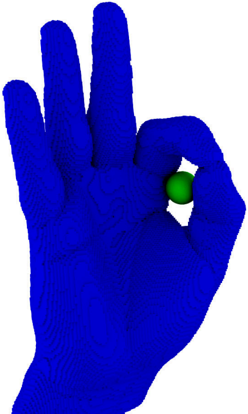
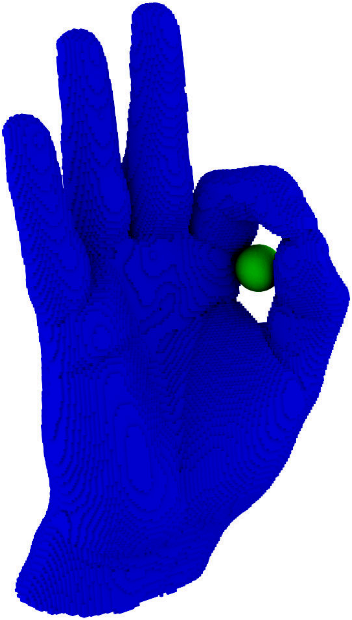




- Breadth ball
- Homology generator
- Close hole



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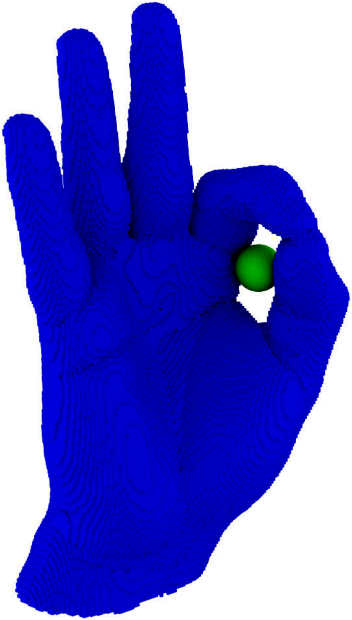
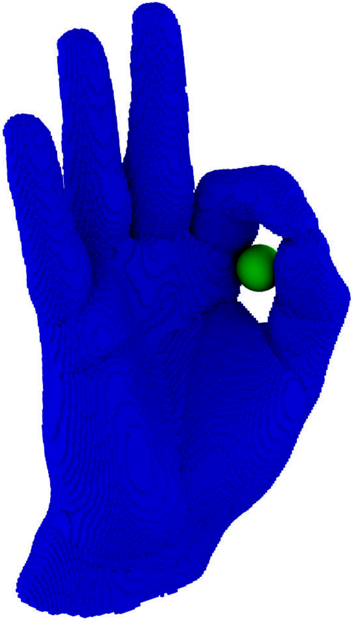
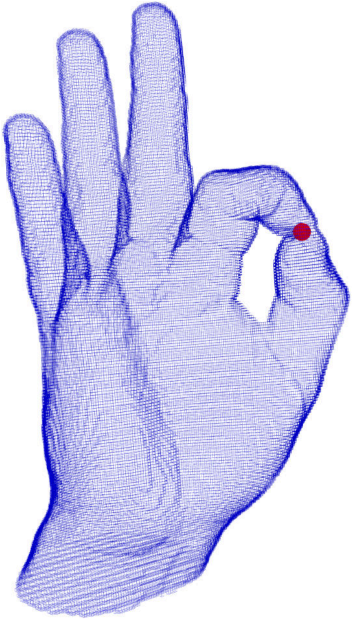


- Breadth ball
- Homology generator
- Close hole





- Thickness ball
- Cohomology generator
- Open hole

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(co)homology generators

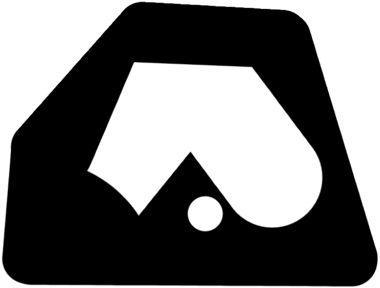
"A good homology generator should be close to a breadth ball"

Algorithms

- Algorithm 6: TB pair $\mapsto$ homology generator
- Algorithm 7: TB pair $\mapsto$ cohomology generator

(co)homology generators

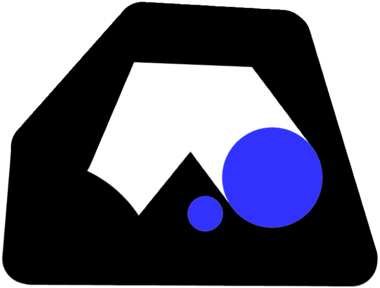
Algorithm 6 (homology generator)



- Discrete object
- Breadth balls
- Filtration

(co)homology generators

Algorithm 6 (homology generator)



- Discrete object
- Breadth balls
- Filtration

(co)homology generators

Algorithm 6 (homology generator)

- Discrete object
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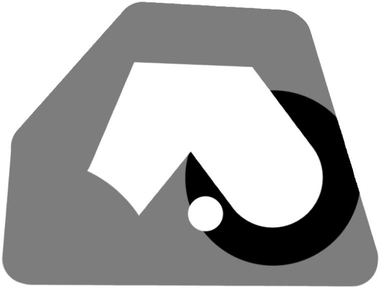
- Filtration

(co)homology generators

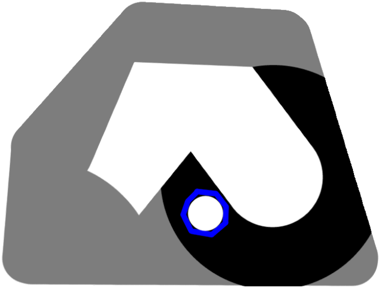
Algorithm 6 (homology generator)



- Discrete object
- Breadth balls
- Filtration

(co)homology generators

Algorithm 6 (homology generator)



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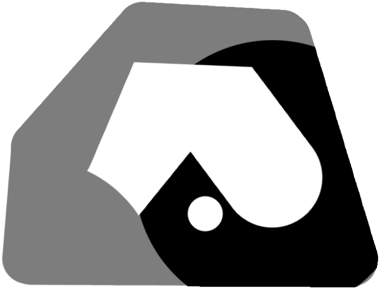

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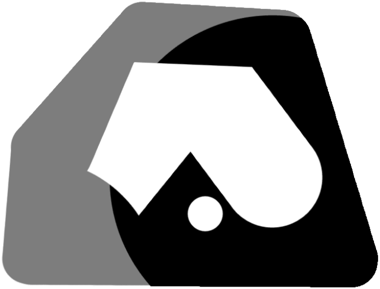

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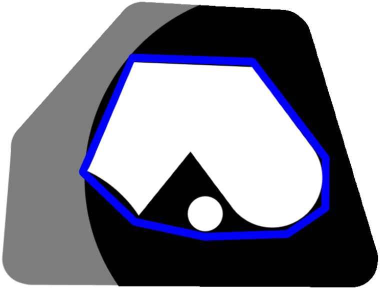

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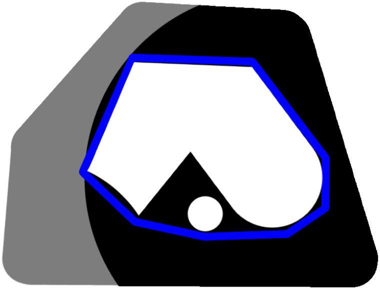

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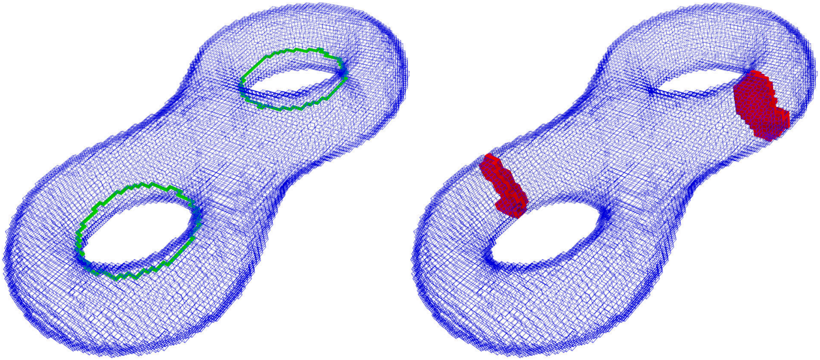
A similar (dual) approach produces **co**homology generators!

(co)homology generators — examples



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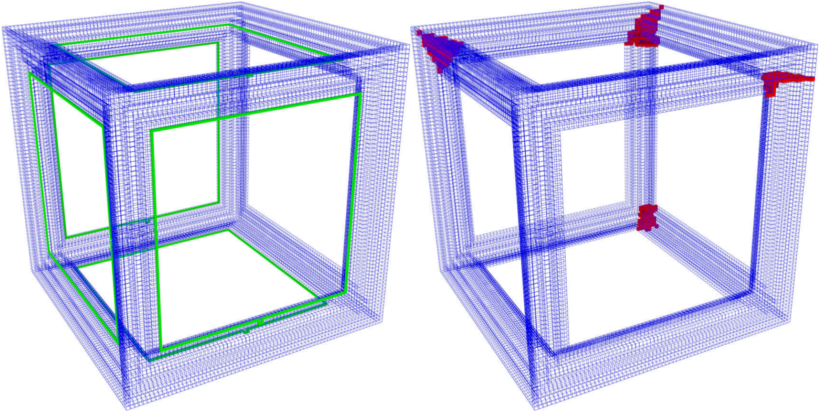

Opening and closing holes

- Thickness balls (and cohomology generators) seem to tell where to **break** a hole
- Breadth balls (and homology generators) seem to tell where to **fill** a hole

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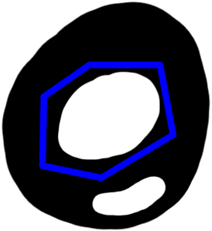
Opening and closing holes

K cubical complex, x cycle, S set of cubes

Opening the cycle x

S opens the cycle x if

1. $K - S$ is a cubical complex
2. $[x] \notin \text{im}(\iota)$
3. $\iota : H(K - S) \to H(K)$ is injective



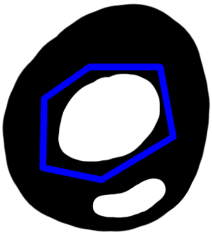
Opening and closing holes

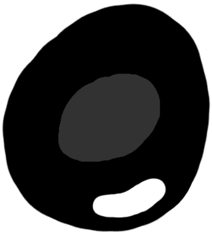
K cubical complex, x cycle, S set of cubes

Closing the cycle x

S closes the cycle x if

1. $K \cup S$ is a cubical complex
2. $[x] \in \ker(\iota)$
3. $\iota : H(K) \to H(K \cup S)$ is surjective



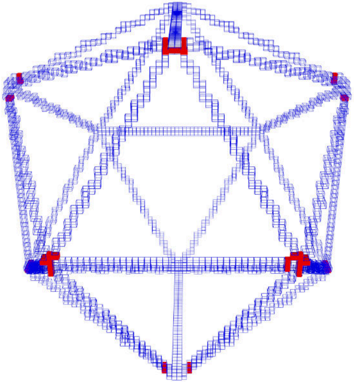


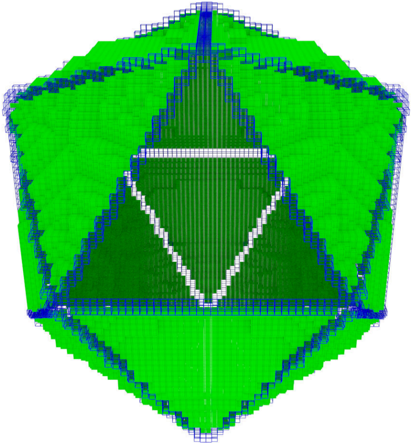
Opening and closing holes

Algorithms

- Algorithm 8: TB pair $\mapsto$ hole opening
- Algorithm 9: TB pair $\mapsto$ hole closing **without surjectivity condition**

Opening and closing holes — examples





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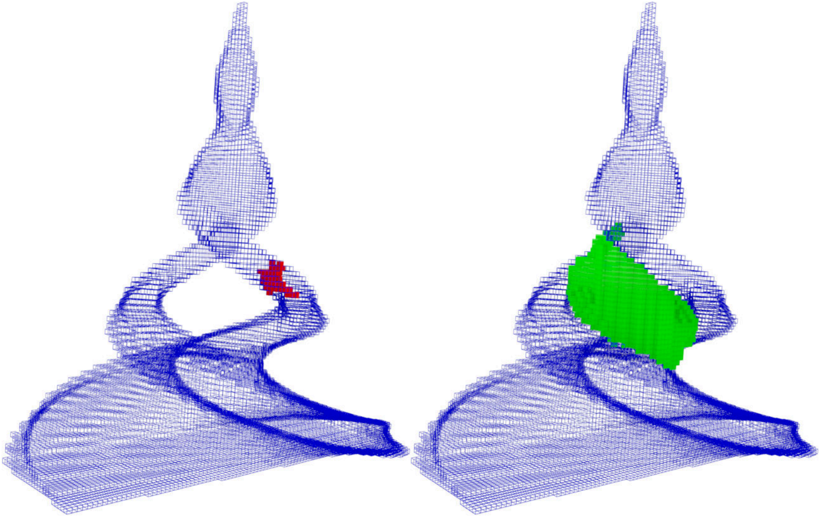

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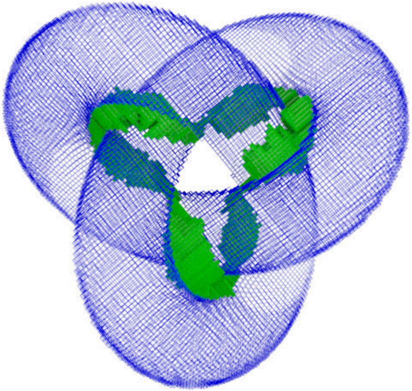

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1 Introduction and Preliminaries

2 The Homological Discrete Vector Field

3 Fast Computation of Betti Numbers on 3D Cubical Complexes

4 Measuring Holes
- Introduction
- Definition
- Thickness-breadth balls
- Applications
- Conclusion

5 Conclusion

Conclusion:

- Topological-geometrical signature of objects
- Robust to noise ⇒ suitable for real applications
- Alternative visualization of holes
- Heuristics for small homology and cohomology generators
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Context

Contributions

Discrete object

↓

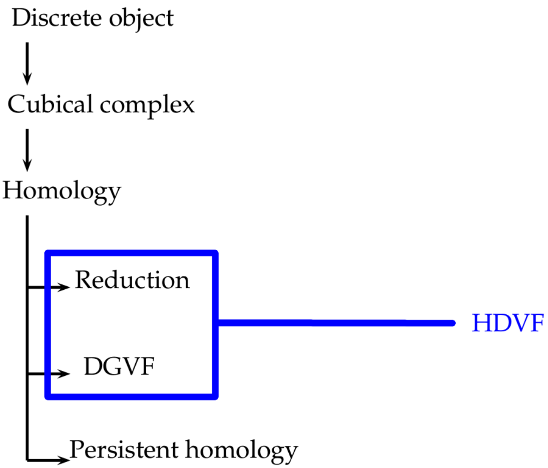
Cubical complex

↓

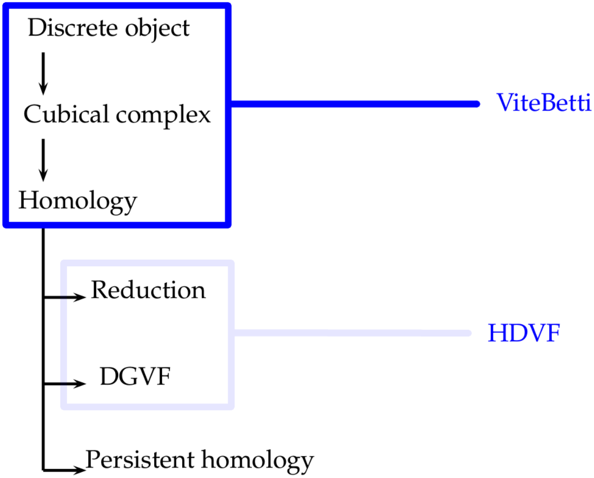
Homology

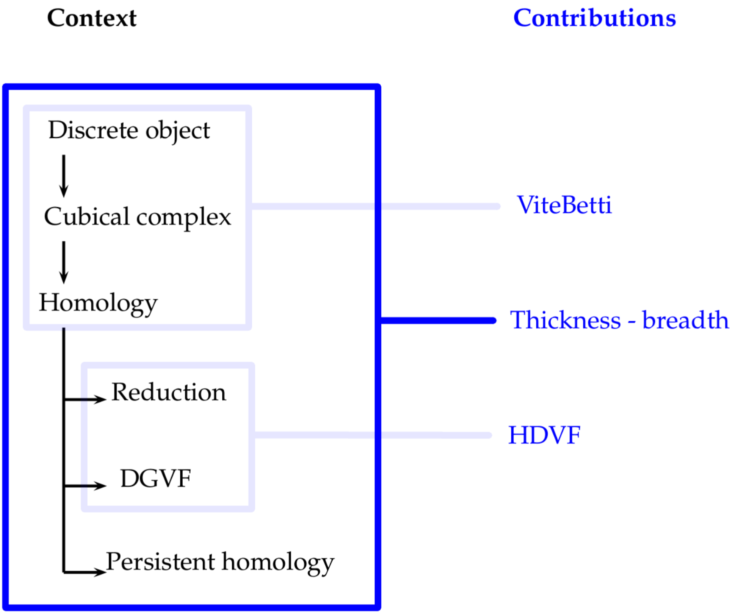
- → Reduction
- → DGVF
- → Persistent homology

Context **Contributions**



Context **Contributions**




Context
Contributions
Discrete object
Cubical complex
Homology
Reduction
DGVF
Persistent homology
ViteBetti
Thickness - breadth
HDVF

Perspectives:

1. HDVF
   - Every CW complex admits a perfect HDVF?
   - Use the operations for comparing HDVFs
   - Compute zigzag persistent homology with HDVFs
2. ViteBetti
   - Apply it directly on the discrete object
   - Try component labeling algorithms from digital geometry context
   - Process complex by slices
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   - To find real world applications
   - To formalize the geometric intuition
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Thanks **Merci** **Grazie** **Gracias**