Mesures géométriques pour les trous dans un objet discret

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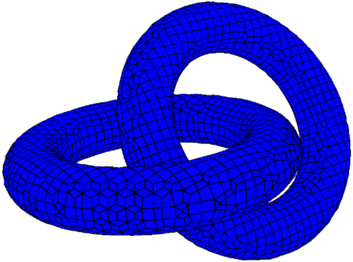
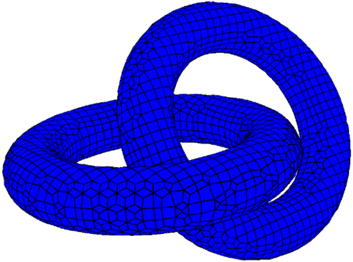
5 mai 2017

Structure



Geometry:

- Volume
- Diameter
- Curvature
- Holes!



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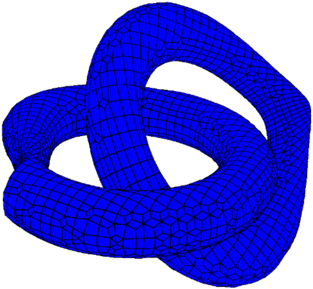

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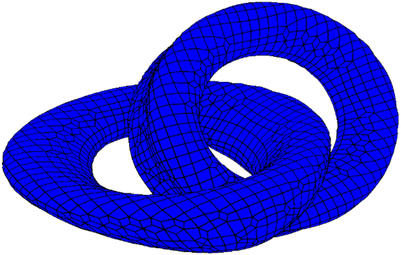

Topology:

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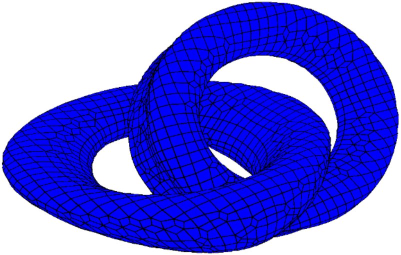

Topology:

- Volume
- Diameter
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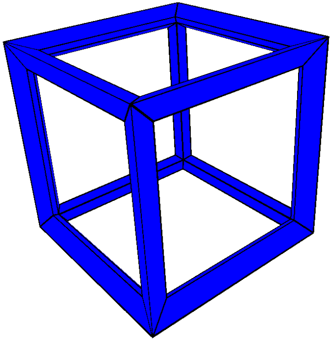
- Holes!

Number of holes: 6? 3?



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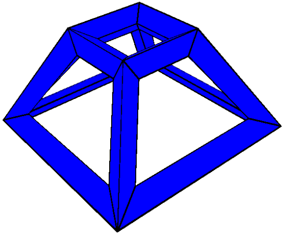
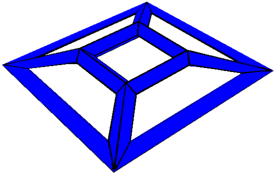
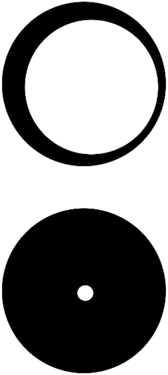


Number of holes: 5

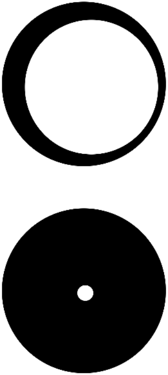


Size of a hole

The 1st one is *bigger* than the 2nd one



The 2nd one is *thicker* than the 1st one



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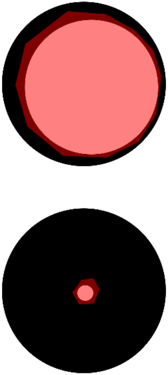

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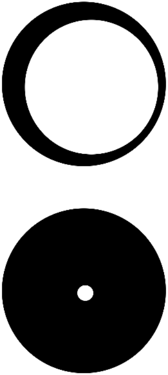

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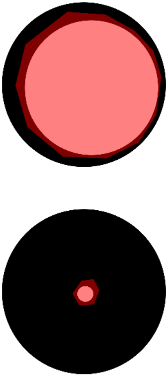

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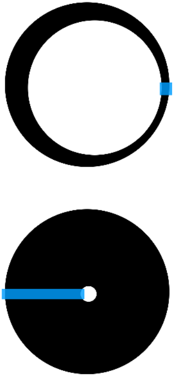
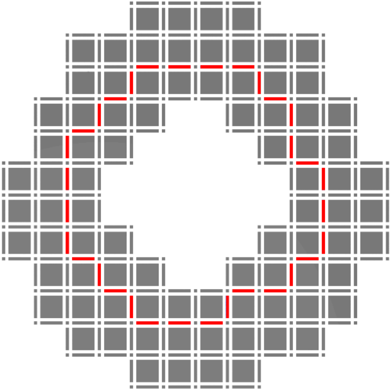
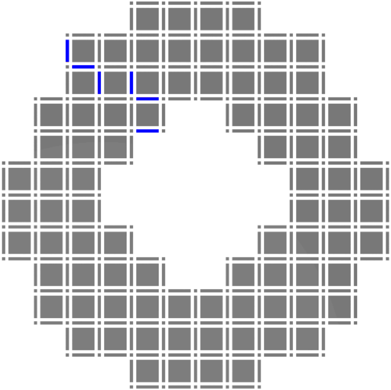


Representing a hole

Homology



Cohomology



Representing a hole

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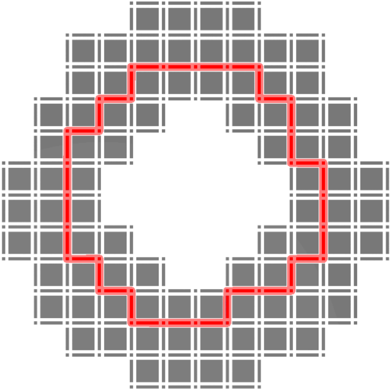

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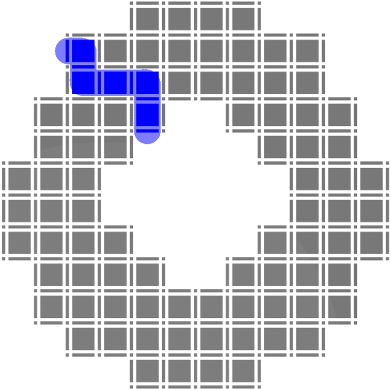
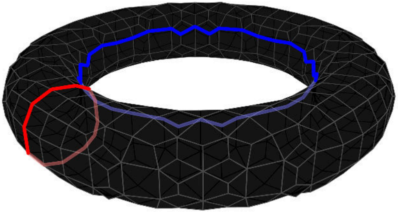
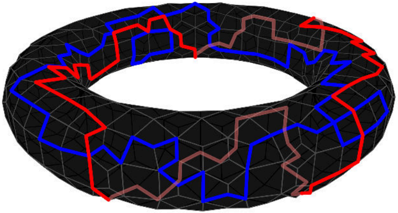


Representing a hole

Do homology generators really represent holes?



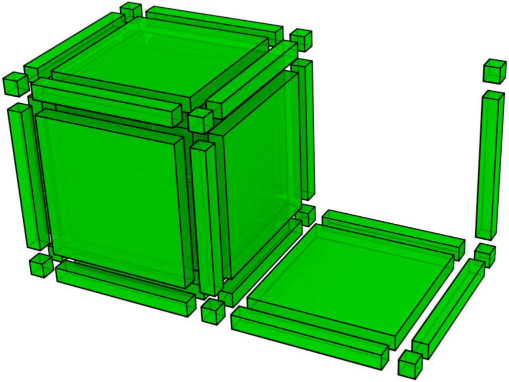


Geometry	+	Topology
$\downarrow$		$\downarrow$
Signed distance transform		Persistent homology

Sections

Cubical complex

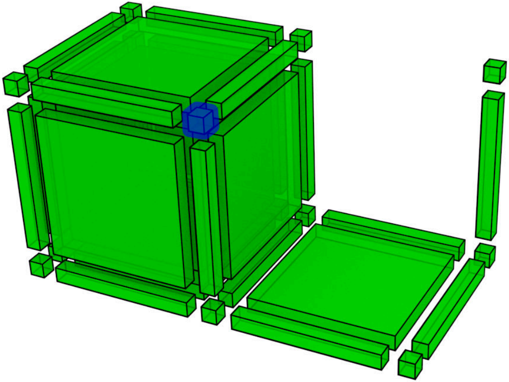
Union[1] of points, edges, squares, cubes, ... (q-dimensional cubes)



[1] with some conditions

Cubical complex

Union[1] of **points**, edges, squares, cubes, ... (q-dimensional cubes)



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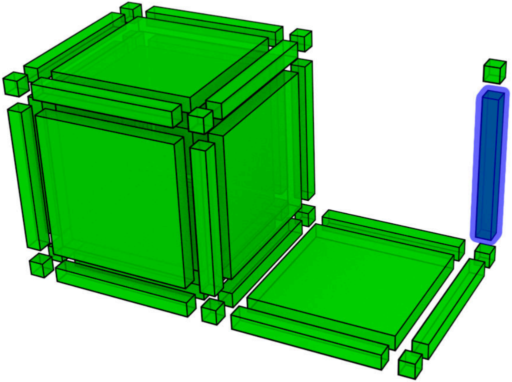

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Cubical complex

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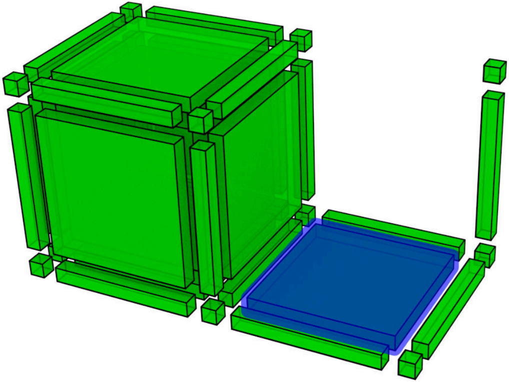
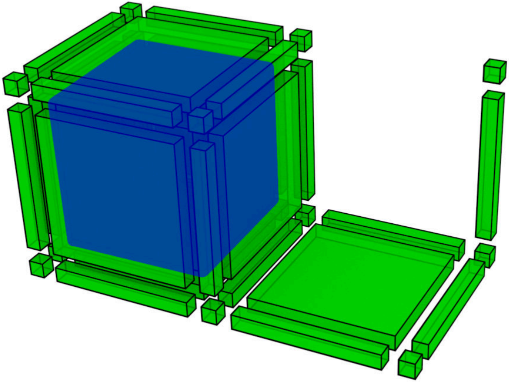


[1] with some conditions

Cubical complex

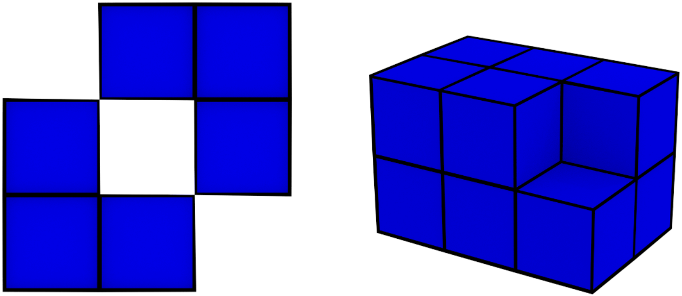
Union[1] of points, edges, squares, **cubes**, ... (*q*-dimensional cubes)



[1] with some conditions

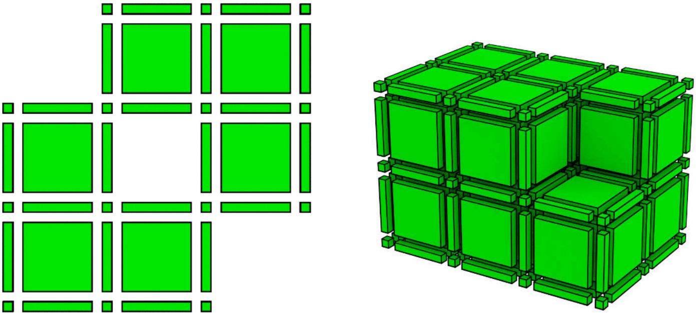
Discrete object

A nD discrete object is a subset of $\mathbb{Z}^n$

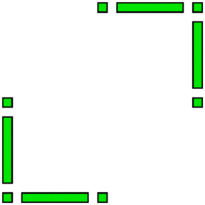


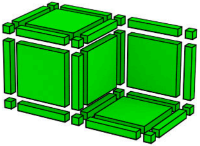
We usually choose a connectivity relation such as the $2n$ or the $(3^n - 1)$-connectivity.

Discrete object + connectivity relation $\longrightarrow$ cubical complex



Discrete object + connectivity relation $\longrightarrow$ cubical complex

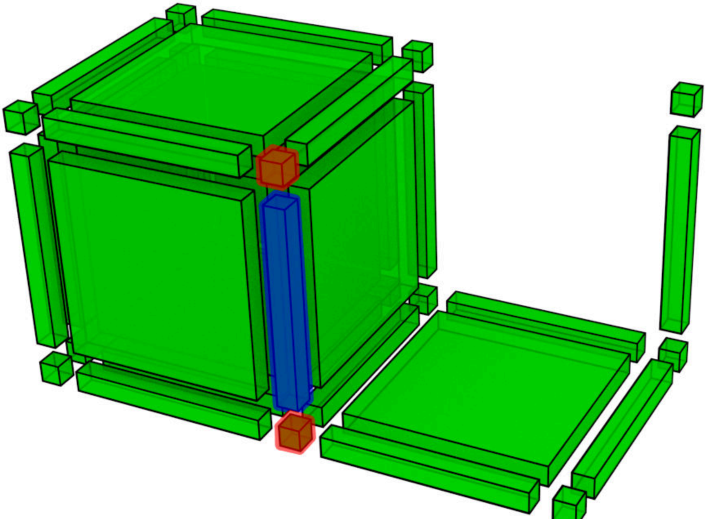




Sections

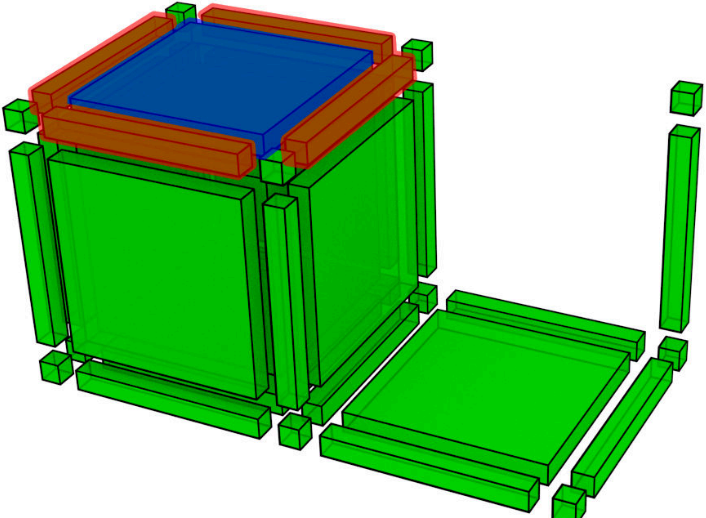
Blue: 1-cube

Red: its boundary (faces)



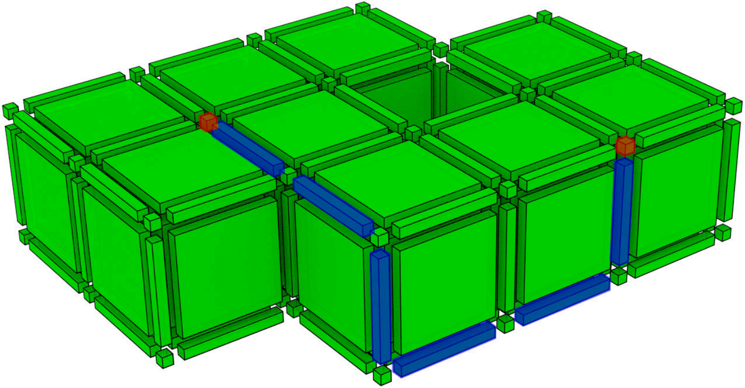
Blue: 2-cube

Red: its boundary (faces)



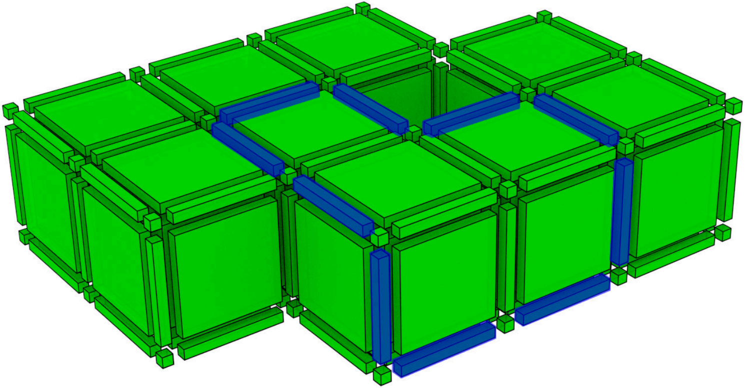
Blue: 1-chain

Red: its boundary



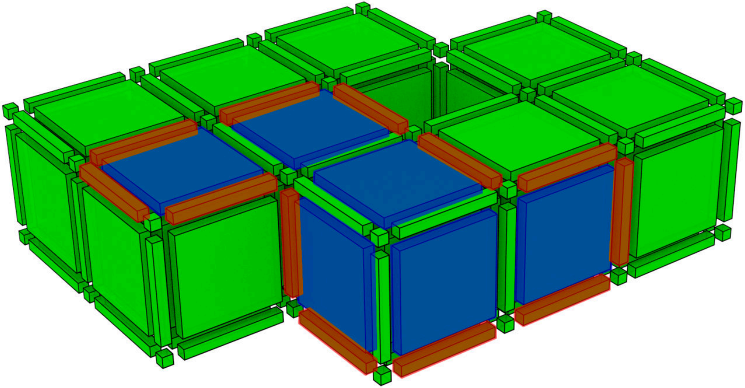
Blue: 1-chain (1-cycle)

Red: its boundary ($= \emptyset$)



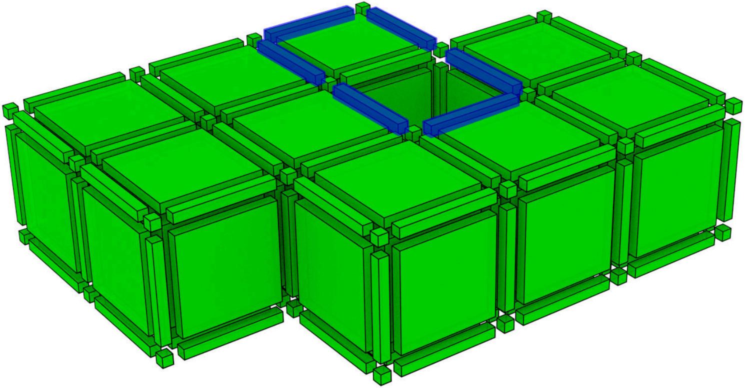
Blue: 2-chain

Red: its boundary (1-cycle)



Blue: 1-chain (1-cycle, but not boundary)

Red: its boundary ($= \emptyset$)



- K cubical complex
- Chain complex of K

$$\cdots C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

where $d_q d_{q+1} = 0 \Rightarrow \operatorname{im}(d_{q+1}) \subset \ker(d_q)$

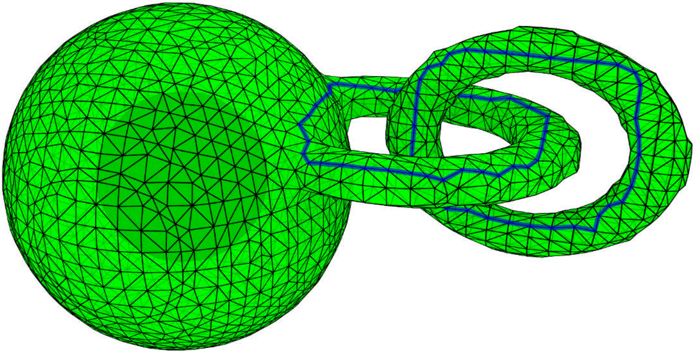
- q-dimensional homology group $H_q(K) := \ker(d_q)/\operatorname{im}(d_{q+1})^2 = (\mathbb{F}_2)^{\beta_q}$
- q-dimensional Betti number: β_q

[2] $\forall x, y \in \ker(d_q), x \sim y \Leftrightarrow x + y \in \operatorname{im}(d_{q+1})$

- $\beta_0 = \#$ connected components (0-holes)
- $\beta_1 = \#$ tunnels or handles (1-holes)
- $\beta_2 = \#$ cavities (2-holes)

Betti numbers are

- Topological invariants $\rightarrow$ classification
- Shape descriptors $\rightarrow$ understanding



$\beta_0 = 2,\ \beta_1 = 2,\ \beta_2 = 1,\ \beta_3 = 0,\ \ldots$

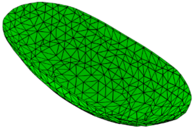
Sections

- Filtration F: $K_1 \subset K_2 \subset K_3 \subset \cdots$

$$\begin{array}{ccccccc}
K_1 & \xrightarrow{\iota} & K_2 & \xrightarrow{\iota} & K_3 & \xrightarrow{\iota} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
H(K_1) & \xrightarrow{\iota_*} & H(K_2) & \xrightarrow{\iota_*} & H(K_3) & \xrightarrow{\iota_*} & \cdots
\end{array}$$

- $\beta_{i,j} = \dim(\iota : H(K_i) \to H(K_j))$
  number of holes in K_i still in K_j
- $\mu_{i,j} = \beta_{i,j} - \beta_{i,j+1} - \beta_{i-1,j} + \beta_{i-1,j+1}$
  number of holes born in K_i and dying in K_j

Example

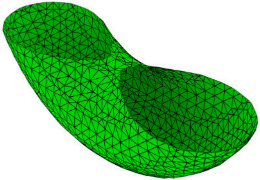


- height: 5
- β_0: **1**
- β_1: 0
- β_2: 0

Persistence intervals:

- Dimension 0:
- Dimension 1:
- Dimension 2:

Example

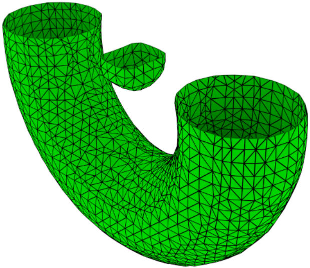


- height: 7
- β_0: 1
- β_1: **1**
- β_2: 0

Persistence intervals:

- Dimension 0:
- Dimension 1:
- Dimension 2:

Example

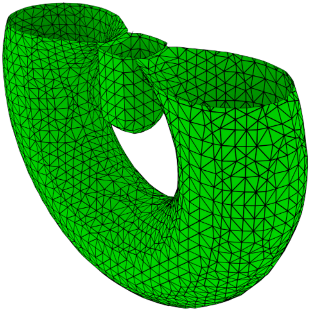


- height: 11
- β_0: **2**
- β_1: 1
- β_2: 0

Persistence intervals:

- Dimension 0:
- Dimension 1:
- Dimension 2:

Example

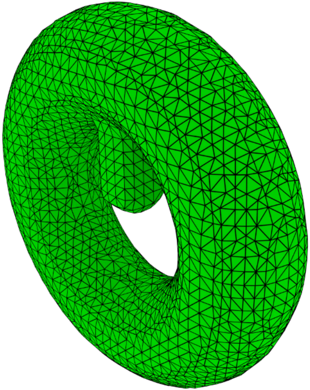


- height: 13
- β_0: **1**
- β_1: **2**
- β_2: 0

Persistence intervals:

- Dimension 0: $(10, 13)$
- Dimension 1:
- Dimension 2:

Example

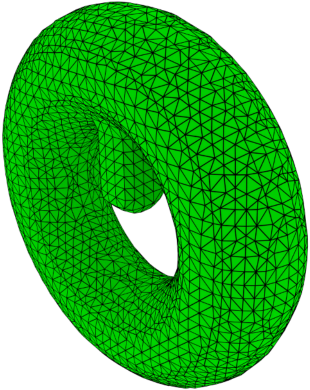


- height: 17
- β_0: 1
- β_1: 2
- β_2: **1**

Persistence intervals:

- Dimension 0: $(10, 13)$
- Dimension 1:
- Dimension 2:

Example



- height:
- β_0: 1
- β_1: 2
- β_2: 1

Persistence intervals:

- Dimension 0: $(10, 13)$, $(0, \infty)$
- Dimension 1: $(7, \infty)$, $(13, \infty)$
- Dimension 2: $(17, \infty)$

Sections

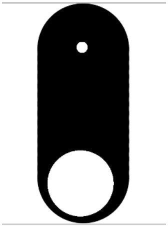
Signed distance transform

Let O be a discrete object,

$$sdt_O(x) = \begin{cases} -d(x, O^c) & \text{if } x \in O \\ d(x, O) & \text{if } x \notin O \end{cases}$$

Figure: Sublevel sets of the signed distance form

Persistent homology with signed distance transform



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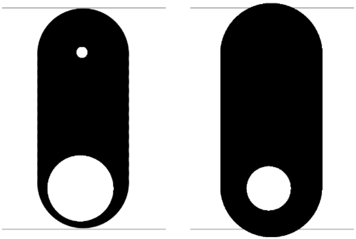

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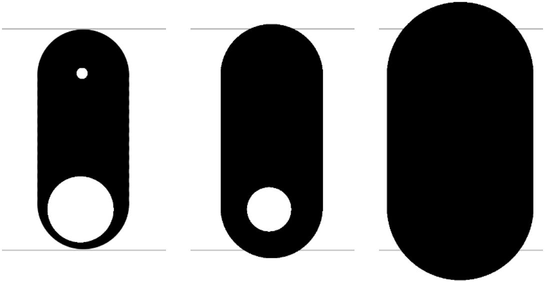

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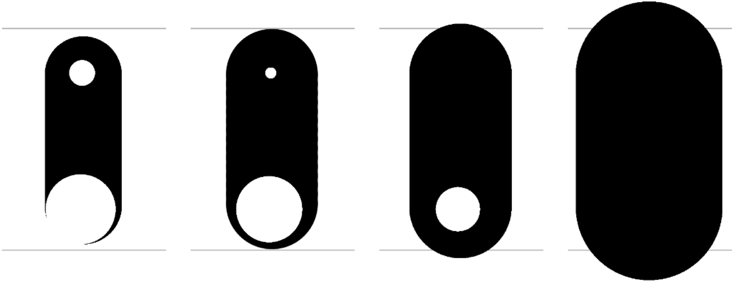
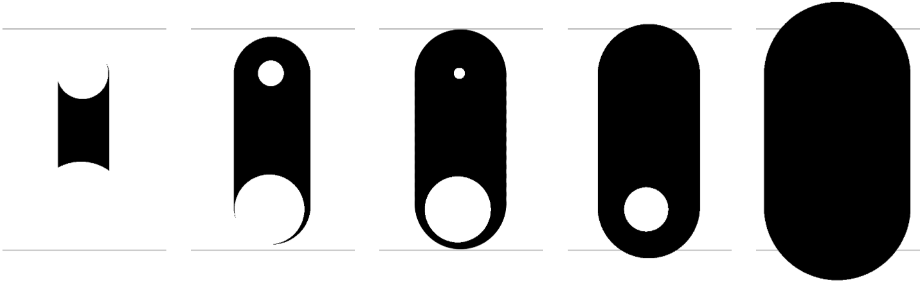
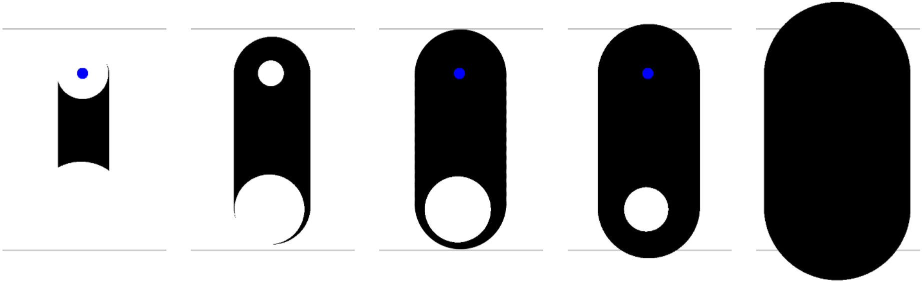


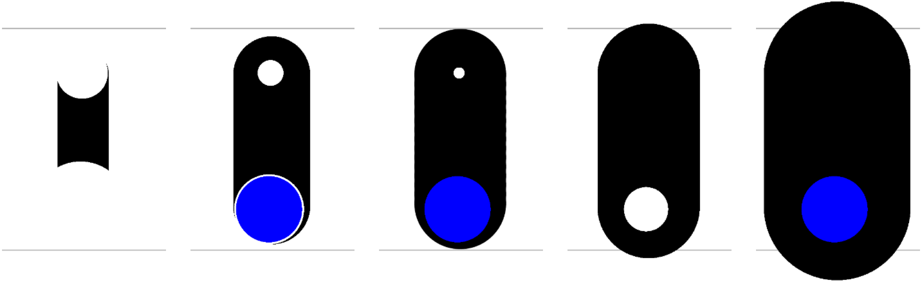
Persistent homology with signed distance transform



Persistent homology with signed distance transform



Persistent homology with signed distance transform



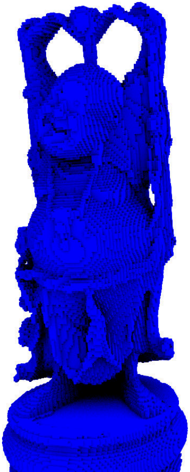
Thickness and breadth

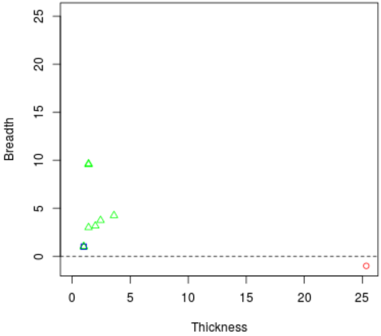
Let O be a discrete object and F the filtration defined by the sublevel sets of its signed distance transform. Let $TB(O) = \{(-x, y) \in PD(F) \mid x \leq 0, y \geq 0\}$. Its intervals are the *thickness-breadth* pairs of O

- There is a thickness-breadth pair (t, b) for each hole of O
- t is the *thickness* of the hole and b, its *breadth*

Thickness-breadth diagram

Thickness-breadth pairs can be represented like persistence diagrams





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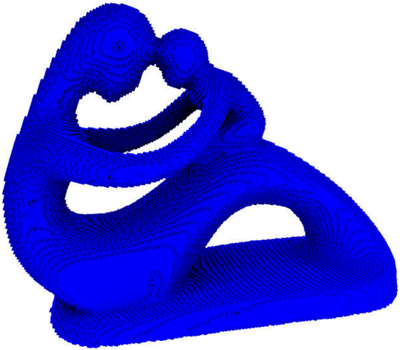

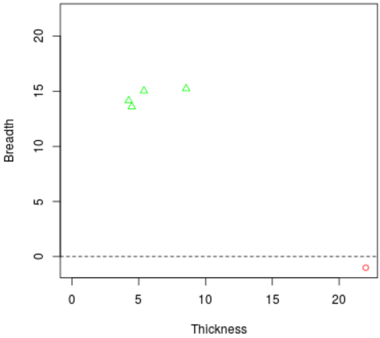

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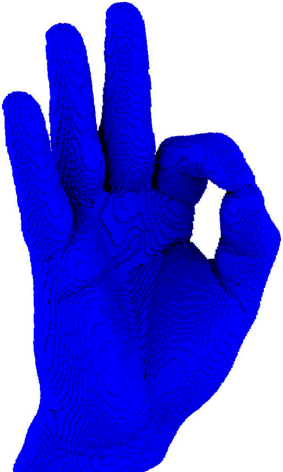

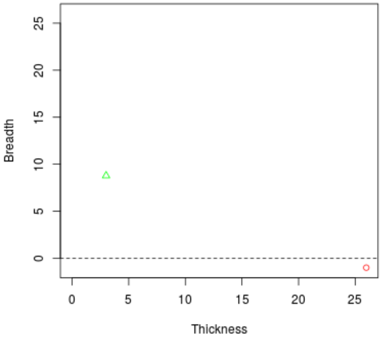

Theorem

Let X and Y be two 3D discrete objects. Let us call

$$\delta = d_H(X, Y) + d_H(\mathbb{Z}^3 \setminus X, \mathbb{Z}^3 \setminus Y) + 2\sqrt{3}$$

Thus, for every thickness-breadth pair $p_X = (x, y)$ of X such that $x, y > \delta$, there exists another thickness-breadth pair $p_Y = (x', y')$ of Y such that

$$||p_X - p_Y||_\infty \leq \delta$$

Sections

Thickness and breadth ball

Let be (t, b) a TB-pair and (σ, τ) its pair of cells

- The *thickness ball* of (t, b) is the ball of radius t centered at σ
- The *breadth ball* of (t, b) is the ball of radius b centered at τ

Thickness and breadth ball

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- The *thickness ball* of (t, b) is the ball of radius t centered at σ
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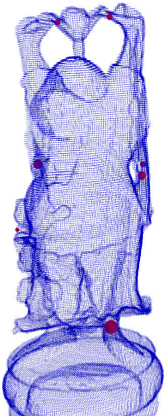
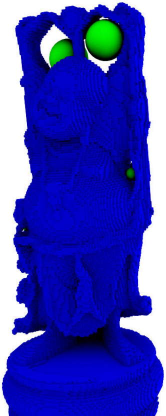
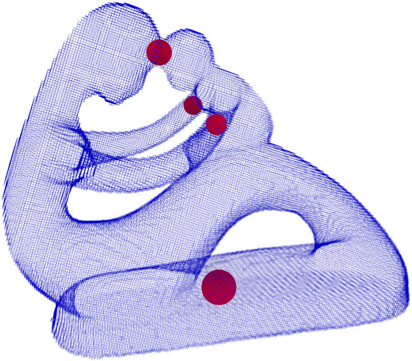


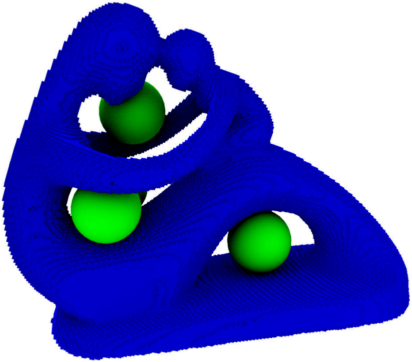


Thickness and breadth ball

Let be (t, b) a TB-pair and (σ, τ) its pair of cells

- The *thickness ball* of (t, b) is the ball of radius t centered at σ
- The *breadth ball* of (t, b) is the ball of radius b centered at τ





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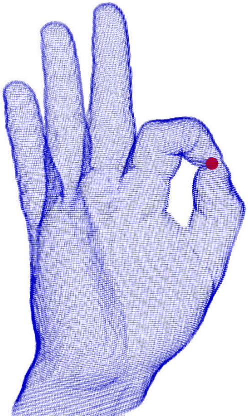
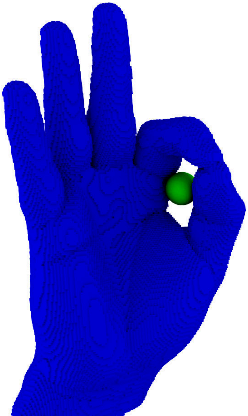
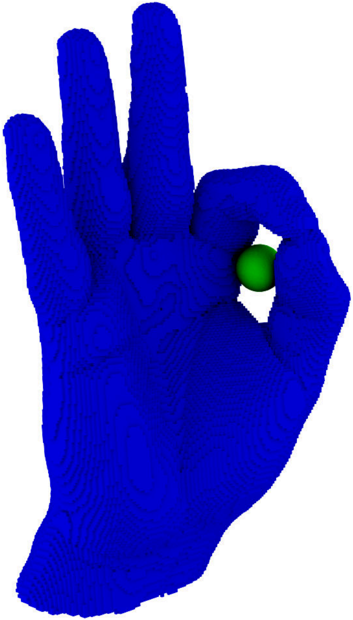




- Breadth ball
- Homology generator
- Close hole



- Breadth ball
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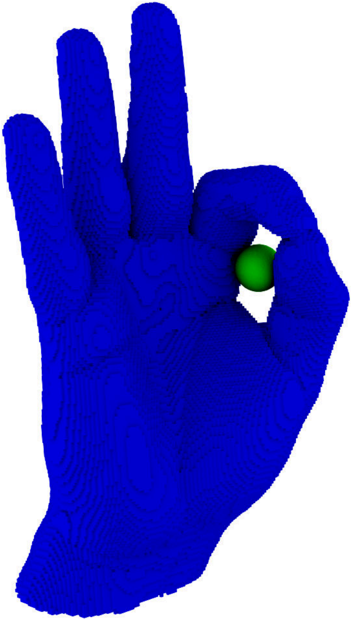
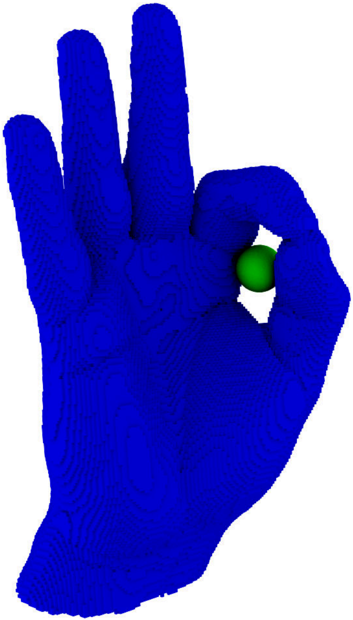
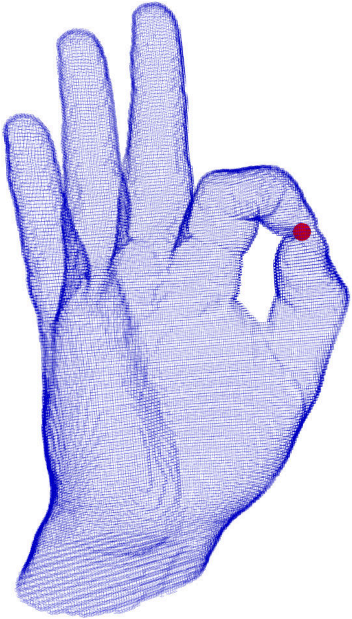


- Breadth ball
- Homology generator
- Close hole





- Thickness ball
- Cohomology generator
- Open hole

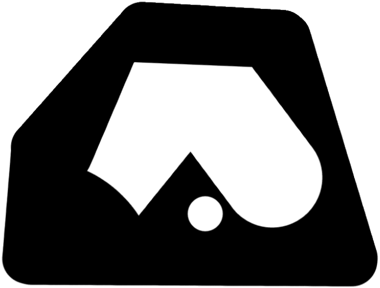
Sections

"A good homology generator should be close to a breadth ball"

Algorithms

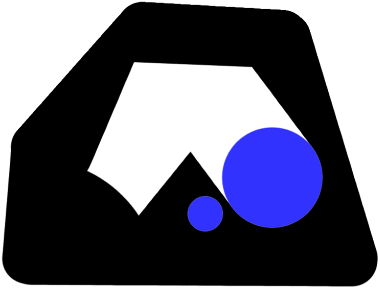
- Algorithm 1: TB pair $\mapsto$ homology generator
- Algorithm 2: TB pair $\mapsto$ cohomology generator

Algorithm 1 (homology generator)



- Discrete object
- Breadth balls
- Filtration

Algorithm 1 (homology generator)



- Discrete object
- Breadth balls
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Algorithm 1 (homology generator)

- Discrete object
- Breadth balls
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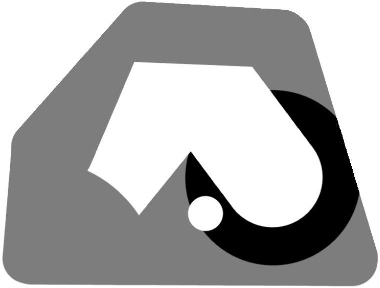

- Discrete object
- Breadth balls
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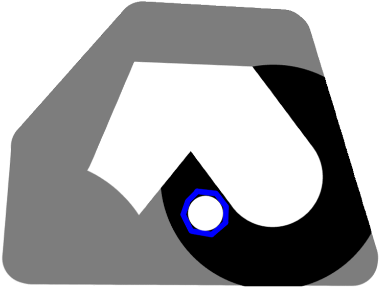
Algorithm 1 (homology generator)



- Discrete object
- Breadth balls
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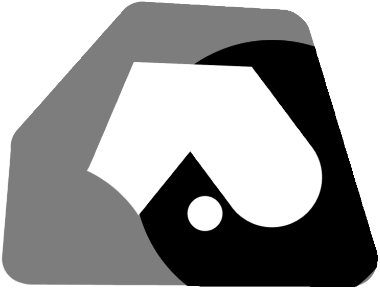
Algorithm 1 (homology generator)



- Discrete object
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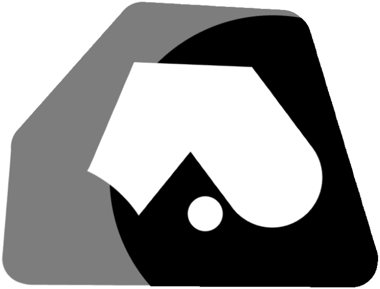
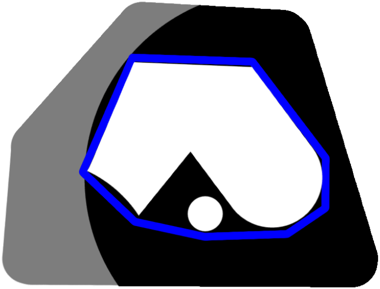
Algorithm 1 (homology generator)



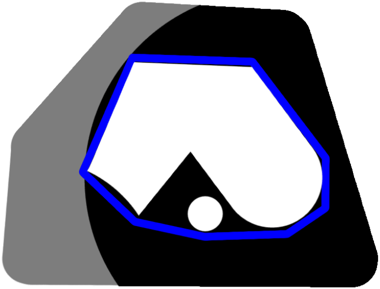
- Discrete object
- Breadth balls
- Filtration

Algorithm 1 (homology generator)



- Discrete object
- Breadth balls
- Filtration

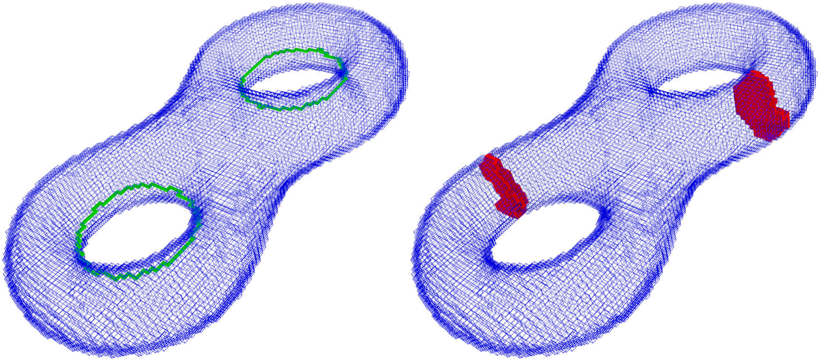
Algorithm 1 (homology generator)



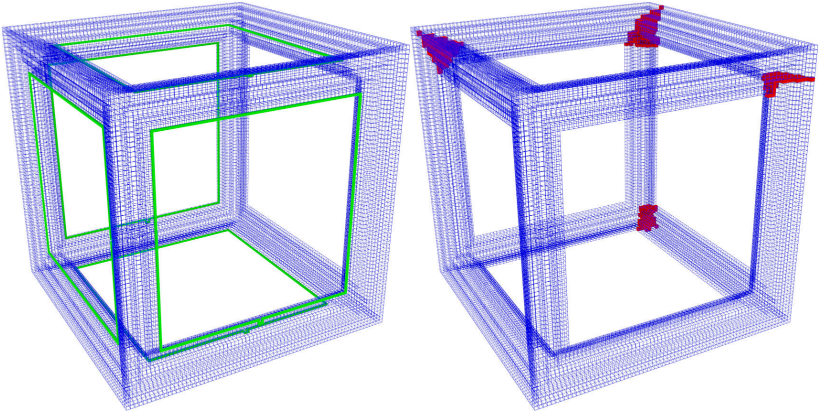
- Discrete object
- Breadth balls
- Filtration

A similar (dual) approach produces **co**homology generators!

Examples



Examples



Conclusion:

- Topological-geometrical signature of objects
- Robust to noise $\rightarrow$ suitable for real applications
- Alternative visualization of holes
- Heuristics for small homology and cohomology generators

You can download this presentation from

http://aldo.gonzalez-lorenzo.perso.luminy.univ-amu.fr/downloads.html

Thanks ! Questions?