TWO MEASURES FOR THE HOMOLOGY GROUPS OF BINARY VOLUMES

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Abstract: Given a binary object (2D or 3D), its Betti numbers characterize the number of holes in each dimension. They are obtained algebraically, and even though they are perfectly defined, there is no unique way to display these holes. We propose two geometric measures for the holes, which are uniquely defined and try to compensate the loss of geometric information during the homology computation: the *thickness* and the *breadth*. They are obtained by filtering the information of the persistent homology computation of a filtration defined through the signed distance transform of the binary object.

Keywords: binary volume, distance transform, persistent homology, geometric information, holes

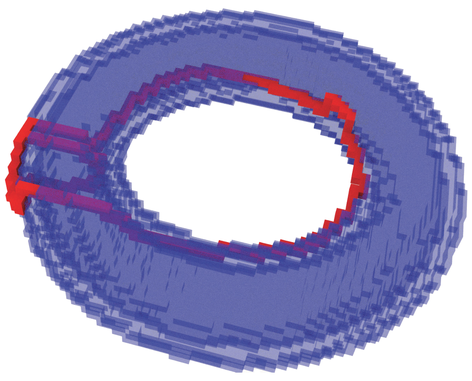
Context

Binary object X (finite subset of Z^n)
↳ Cubical complex K(X)
↳ Homology groups:

$$H_q(X) = \ker(d_q)/\operatorname{im}(d_{q+1})$$

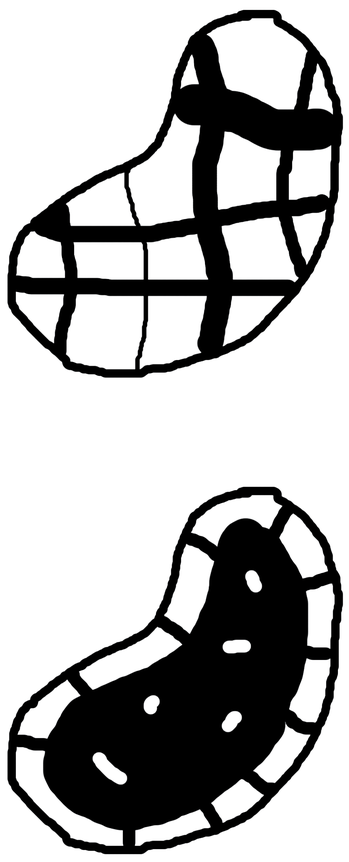
Betti numbers: β_q = dimension of $H_q(X)$. They are the number of holes of each dimension

Homology generators: representatives of some base of the homology groups



Problem

Homology allows to define the number of holes in an object, but the geometry is neglected. We cannot determine the "size" of a hole. Also, its representation is ambiguous

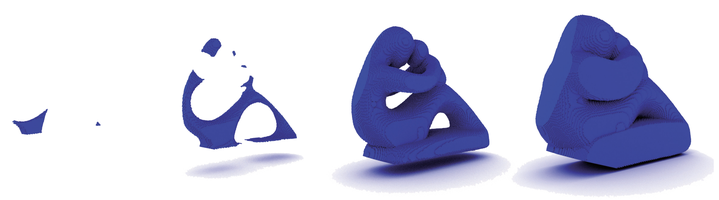


Measures

We need to combine geometry and homology:

1. Signed distance transform

$$sdt_X(x) = \begin{cases} -d(x,X) = -\min\{d(x,y) \mid y \notin X\} & \text{if } x \in X \\ d(x,\mathbb{Z}^3 \setminus X) = \min\{d(x,y) \mid y \in X\} & \text{if } x \notin X \end{cases}$$



2. Persistent homology

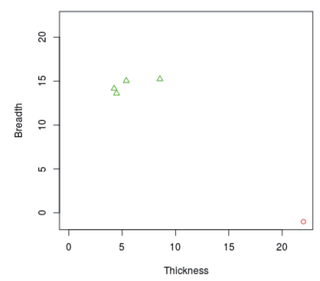
It defines the birth and death times of the holes of a sequence of nested complexes (filtration) as a set of intervals *P*

Thickness and breadth:

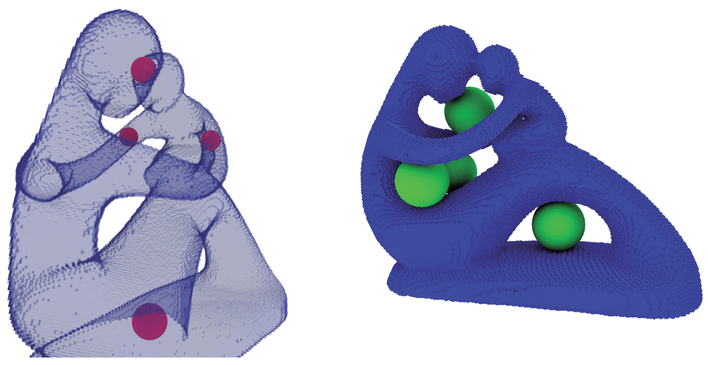
Let *P* be the set of persistent intervals of the filtration induced by the signed distance transform of a binary object. Thus

- The thickness of the holes of X are the values *x* such that *-x* is the left endpoint of an interval of *P* containing the 0
- The breadth of the holes of X are the values *y* such that *y* is the right endpoint of an interval of *P* containing the 0

They can be represented via the thickness-breadth diagram



We can also visualize these measures in terms of balls, which seem to nicely represent where the holes are



Conclusion

- Definition of two geometric measures for holes
- Alternative visualization of holes
- Any dimension
- Useful for classification or understanding

Future works

- Simplicial complexes
- Real world applications: medical context, geostatistics, etc.
- Small generators of homology or cohomology

- Opening and closing of holes

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