Fast computation of Betti numbers on three-dimensional cubical complexes

Aldo González Lorenzo[1,2], Jean-Luc Mari[1], Alexandra Bac[1]
Pedro Real[2]

[1]Aix-Marseille Université, CNRS, LSIS UMR 7296 (France)

[2]Universidad de Sevilla, IMUS (Spain)

20 july 2015

Structure

Structure

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Structure

Union of points, segments, squares and cubes embedded in $\mathbb{Z}^3$.

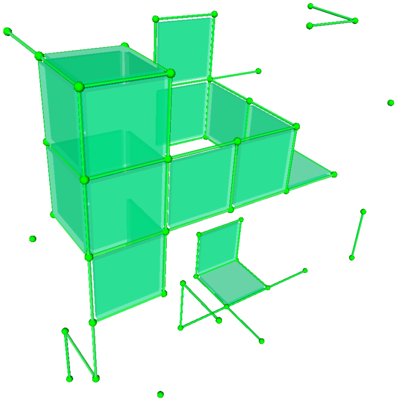


Figure : A 3D cubical complex

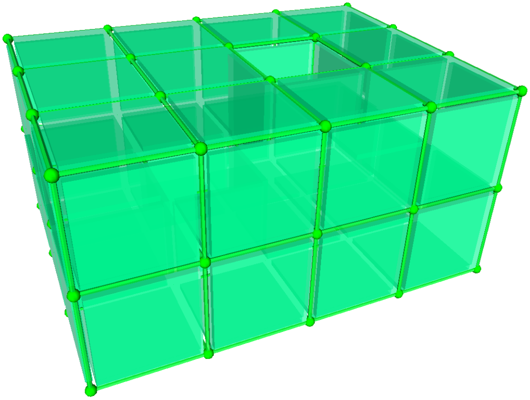
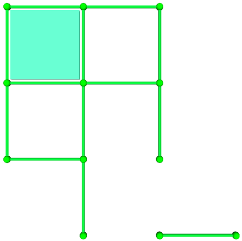


Figure : The Bing's house

A 3D cubical complex can be represented by a binary three-dimensional array



1	1	1	1	1	0	0
1	1	1	0	1	0	0
1	1	1	1	1	0	0
1	0	1	0	1	0	0
1	1	1	0	1	0	0
0	0	1	0	0	0	0
0	0	1	0	1	1	1

Figure : Left: a 2D cubical complex. Right: its matrix representation

Cubical complex $\rightarrow$ Chain complex $\rightarrow$ Homology groups:

$$H_0, H_1, H_2, \ldots$$

The Betti numbers are the ranks of these groups:

$$\beta_0 = \text{rank}(H_0), \beta_1 = \text{rank}(H_1), \beta_2 = \text{rank}(H_2), \ldots$$

The Betti numbers give a mathematical definition of the number of connected components (β_0), tunnels (β_1) and voids (β_2) in a space.

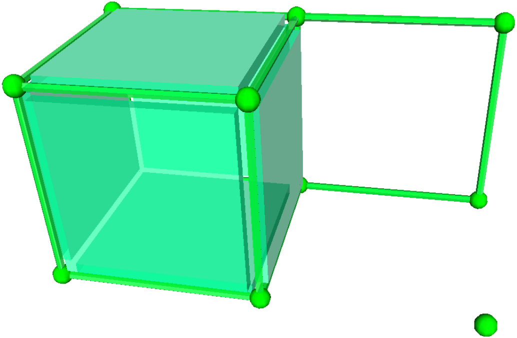


Figure : $\beta_0 = 2$, $\beta_1 = 1$, $\beta_2 = 1$

- $\beta_0(K) =$ number of connected components of K
- Euler-Poincaré characteristic:

$$\chi(K) = |K^0| - |K^1| + |K^2| - |K^3| = \beta_0 - \beta_1 + \beta_2$$

- Alexander duality:

$$H_q(K) \cong H^{2-q}(\mathbb{S}^3 \setminus K) \cong H_{2-q}(\mathbb{S}^3 \setminus K) \text{ (reduced homology)}$$

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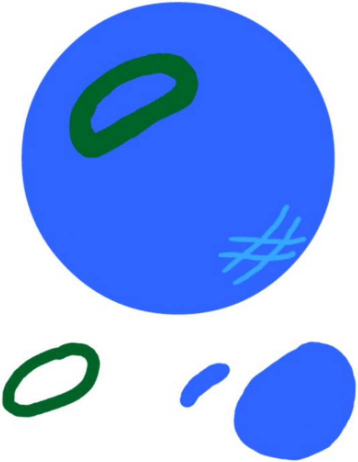


$$K : (0, 1)$$
$$\mathbb{S}^2 \setminus K : (1, 0)$$

Therefore,

- $\beta_0(K) =$ number of connected components of K
- $\beta_2(K) =$ number of connected components of $\mathbb{S}^3 \setminus K - 1$
- $\beta_1(K) = \beta_0 + \beta_2 - \chi(K)$

Require: A 3D cubical complex K
Ensure: $\beta_0, \beta_1, \beta_2$

Count the number of connected components of K together with $\chi(K)$

Count the number of connected components of $\bar{K}$ except for those touching the border of the array

1	1	1	1	1	0	0
1	1	1	0	1	0	0
1	1	1	1	1	0	0
1	0	1	0	1	0	0
1	1	1	0	1	0	0
0	0	1	0	0	0	0
0	0	1	0	1	1	1

$$\beta_0 = 0$$
$$\chi = 0$$
$$\beta_2 = 0$$

1	1	1	1	1	0	0
1	1	1	0	1	0	0
1	1	1	1	1	0	0
1	0	1	0	1	0	0
1	1	1	0	1	0	0
0	0	1	0	0	0	0
0	0	1	0	1	1	1

$\beta_0 = \mathbf{1}$

$\chi = \mathbf{-1}$

$\beta_2 = 0$

1	1	1	1	1	0	0
1	1	1	0	1	0	0
1	1	1	1	1	0	0
1	0	1	0	1	0	0
1	1	1	0	1	0	0
0	0	1	0	0	0	0
0	0	1	0	1	1	1

$$\beta_0 = \mathbf{2}$$

$$\chi = \mathbf{0}$$

$$\beta_2 = 0$$

1	1	1	1	1	0	0
1	1	1	0	1	0	0
1	1	1	1	1	0	0
1	0	1	0	1	0	0
1	1	1	0	1	0	0
0	0	1	0	0	0	0
0	0	1	0	1	1	1

$$\beta_0 = 2$$
$$\chi = 0$$
$$\beta_2 = \mathbf{0}$$

1	1	1	1	1	0	0
1	1	1	0	1	0	0
1	1	1	1	1	0	0
1	0	1	0	1	0	0
1	1	1	0	1	0	0
0	0	1	0	0	0	0
0	0	1	0	1	1	1

$$\beta_0 = 2$$
$$\chi = 0$$
$$\beta_2 = \mathbf{1}$$

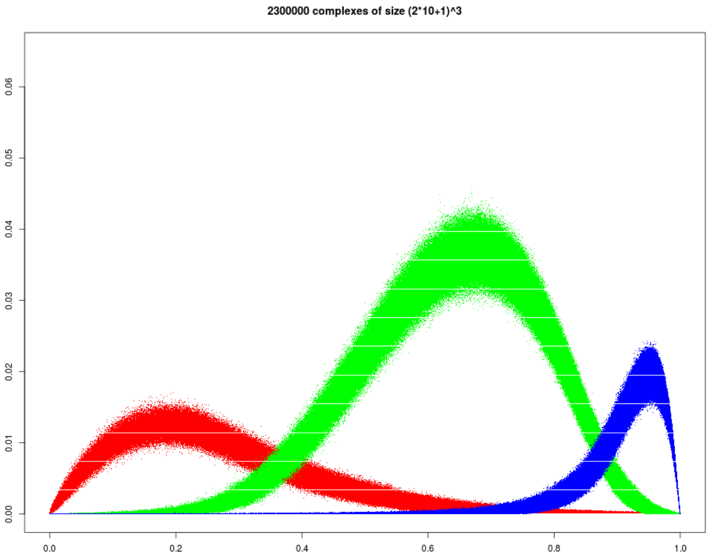
1	1	1	1	1	0	0
1	1	1	0	1	0	0
1	1	1	1	1	0	0
1	0	1	0	1	0	0
1	1	1	0	1	0	0
0	0	1	0	0	0	0
0	0	1	0	1	1	1

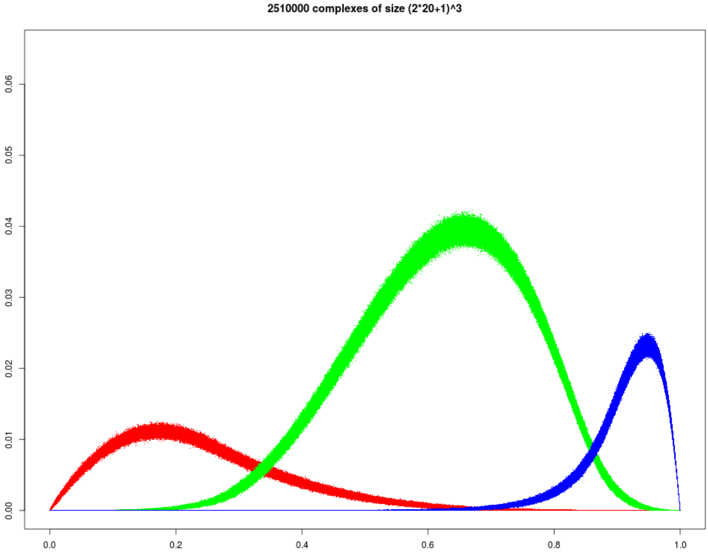
$$\beta_0 = 2$$
$$\chi = 0$$
$$\beta_2 = \mathbf{2}$$

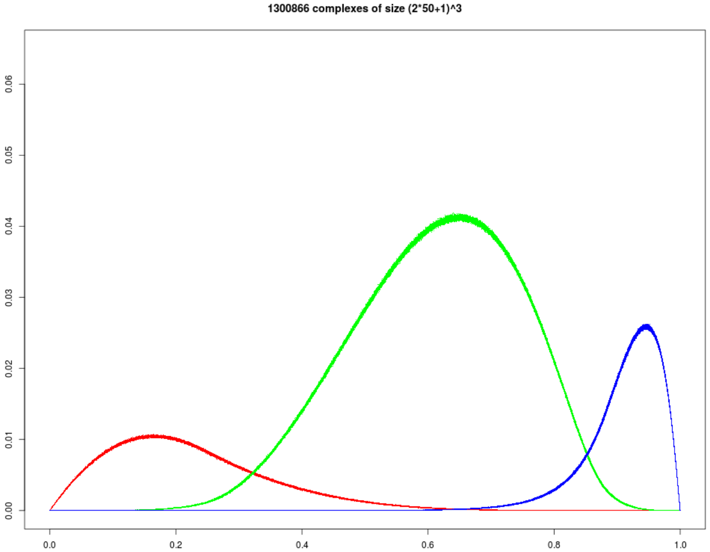
1	1	1	1	1	0	0
1	1	1	0	1	0	0
1	1	1	1	1	0	0
1	0	1	0	1	0	0
1	1	1	0	1	0	0
0	0	1	0	0	0	0
0	0	1	0	1	1	1

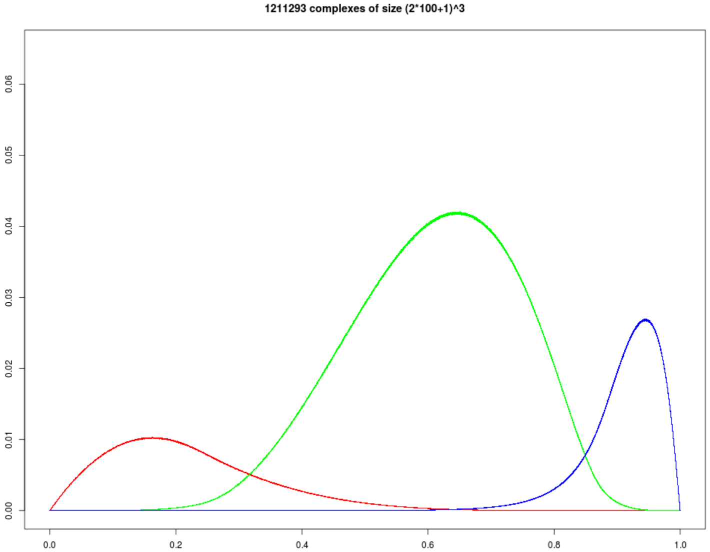
$$\beta_0 = 2$$
$$\chi = 0$$
$$\beta_2 = \mathbf{2}$$

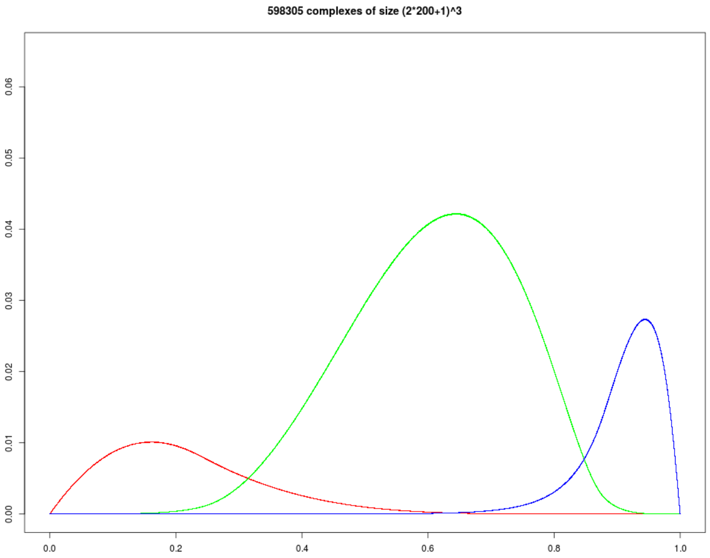
Require: $m \in \mathbb{Z}$, $p \in [0, 1]$
Ensure: A random cubical complex embedded in $[0, m]^3$ with probability p
  for all $\sigma \in [0, m]^3$ **do**
    if rand() $< p$ **then**
      $K \leftarrow \sigma$
    end if
  end for

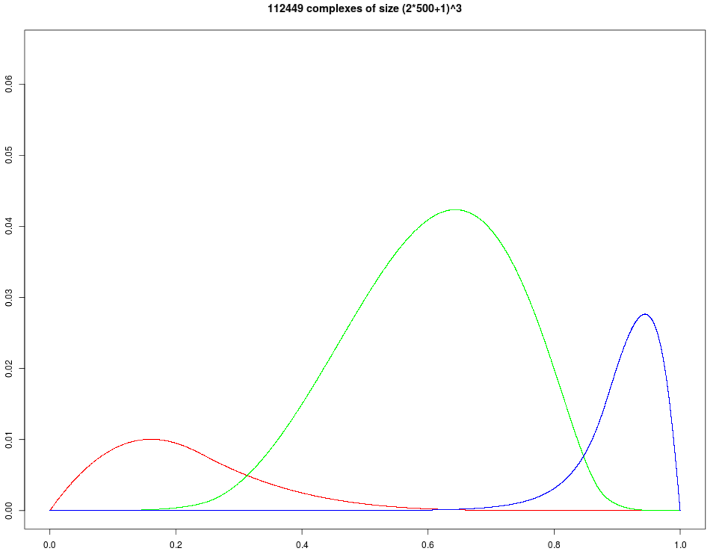

2300000 complexes of size (2*10+1)^3
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.0
0.2
0.4
0.6
0.8
1.0

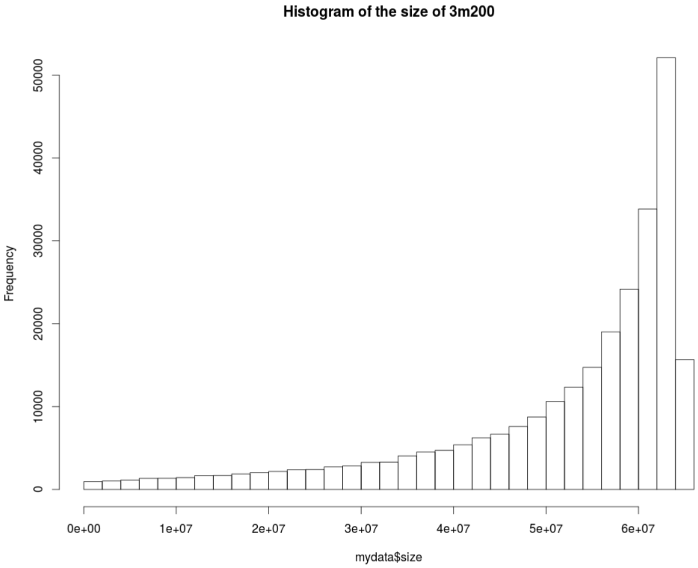

2510000 complexes of size (2*20+1)^3
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.0
0.2
0.4
0.6
0.8
1.0


1300866 complexes of size (2*50+1)^3
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.0
0.2
0.4
0.6
0.8
1.0


1211293 complexes of size (2*100+1)^3
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.0
0.2
0.4
0.6
0.8
1.0

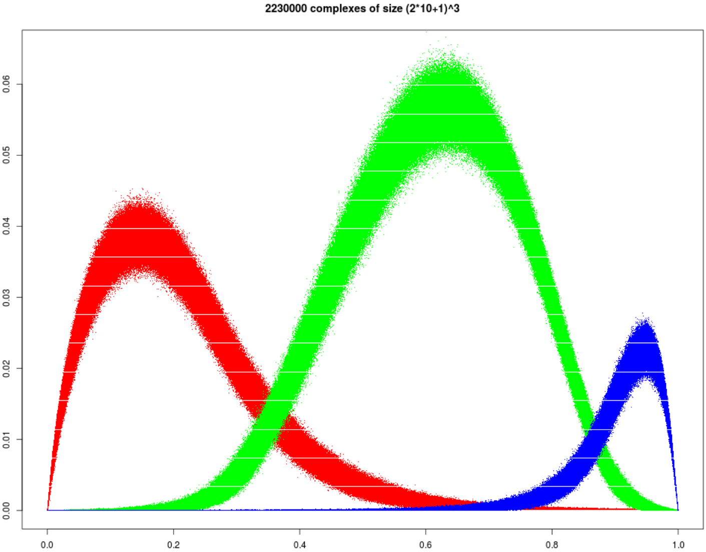

598305 complexes of size (2*200+1)^3
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.0
0.2
0.4
0.6
0.8
1.0

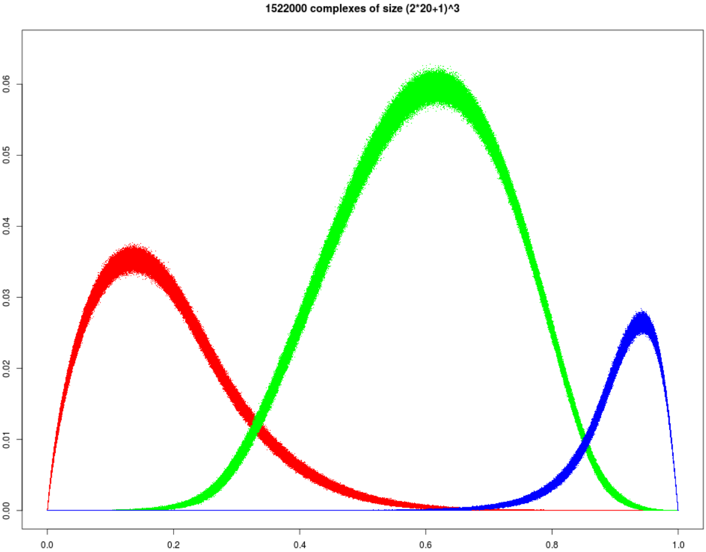

112449 complexes of size (2*500+1)^3
0.06
0.05
0.04
0.03
0.02
0.01
0.00
0.0
0.2
0.4
0.6
0.8
1.0

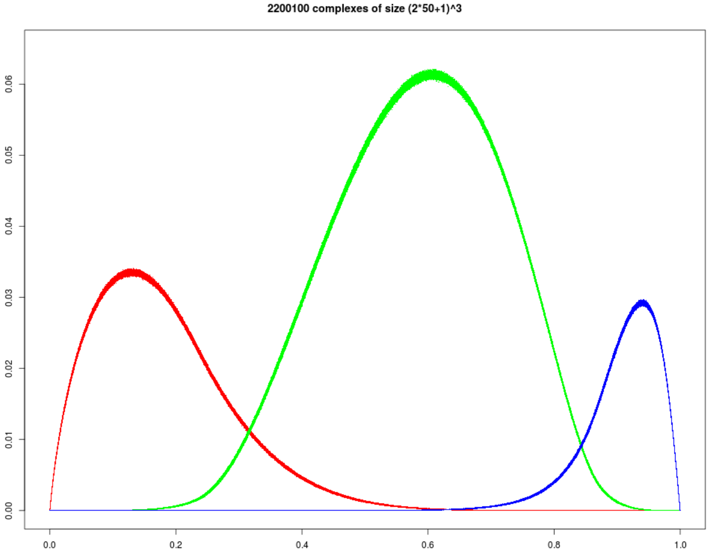

Histogram of the size of 3m200
Frequency
0
10000
20000
30000
40000
50000
0e+00
1e+07
2e+07
3e+07
4e+07
5e+07
6e+07
mydata$size

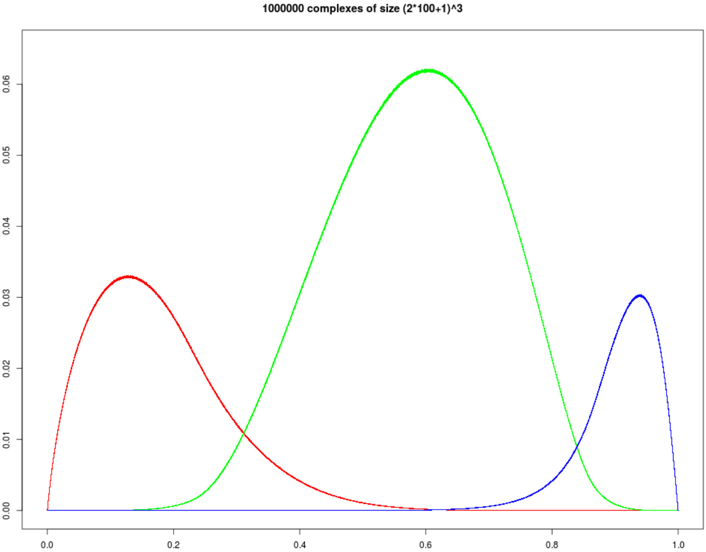
Require: $m \in \mathbb{Z}$, $p \in [0,1]$
Ensure: A random cubical complex embedded in $[0,m]^3$ with probability p

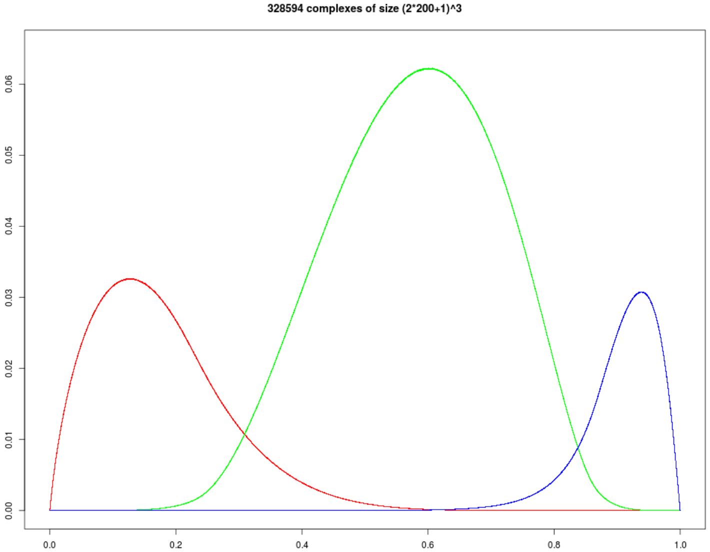
```
for all σ ∈ [0, m]^3 do
    if rand() < p^(dim(σ)+1) then
        K ← σ
    end if
end for
```

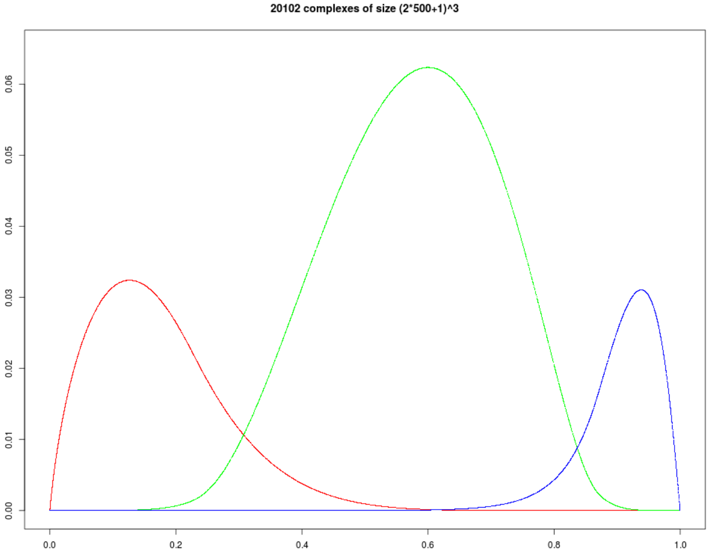

2230000 complexes of size (2*10+1)^3
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.0
0.2
0.4
0.6
0.8
1.0

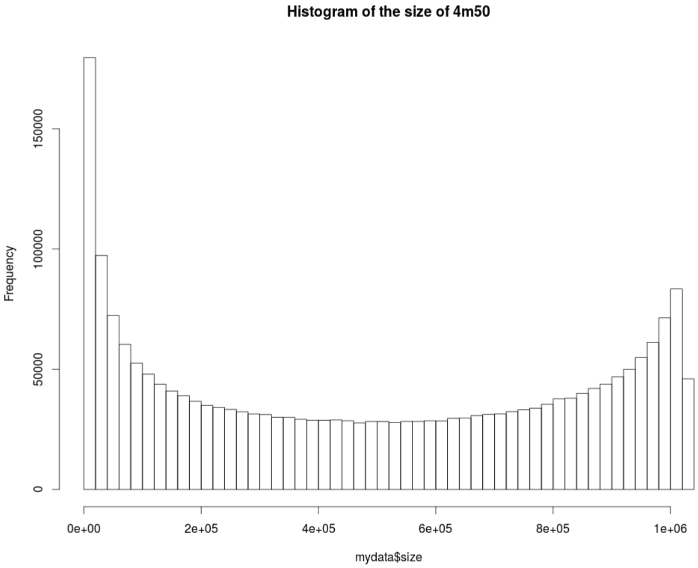

1522000 complexes of size (2*20+1)^3
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.0
0.2
0.4
0.6
0.8
1.0


2200100 complexes of size (2*50+1)^3
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.0
0.2
0.4
0.6
0.8
1.0


1000000 complexes of size (2*100+1)^3
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.0
0.2
0.4
0.6
0.8
1.0


328594 complexes of size (2*200+1)^3
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.0
0.2
0.4
0.6
0.8
1.0


20102 complexes of size (2*500+1)^3
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.0
0.2
0.4
0.6
0.8
1.0


Histogram of the size of 4m50
Frequency
0
50000
100000
150000
0e+00
2e+05
4e+05
6e+05
8e+05
1e+06
mydata$size

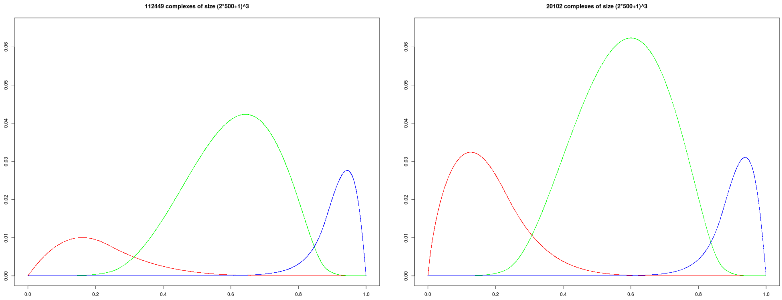


Figure : Comparison between the two random complexes

Size complex (n)	Time (sec)	$1.5 \cdot 10^{-7} n$
$(2 \cdot 10 + 1)^3$	0.0009	0.0013
$(2 \cdot 20 + 1)^3$	0.0068	0.0103
$(2 \cdot 50 + 1)^3$	0.1483	0.1545
$(2 \cdot 100 + 1)^3$	1.2735	1.2180
$(2 \cdot 200 + 1)^3$	11.4484	9.6721
$(2 \cdot 500 + 1)^3$	246.9240	150.4505
$(2 \cdot 1000 + 1)^3$	?	1201.801

- Compute only the Betti numbers for 3D cubical complexes
- Linear time complexity in the size of the bounding box
- Not generalizable to higher dimensions
- Parallel version in process

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Thank you. Questions?