Some news about coverability

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SDA2 Days, 4th July 2016

Welcome to coverability

- $\Sigma =$ finite alphabet
- $\mathbf{w} \in \Sigma^{\mathbb{N}}$
- **Factor:** finite block of $\mathbf{w}$
- **Special factor $\mathbf{u}$:** $u \cdot a$ and $u \cdot b$ occur in $\mathbf{w}$

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- **$\mathbf{q}$-Coverable:** each position of $\mathbf{w}$ is in an occurrence of q

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(Cf. Apostolico, Ehrenfeucht 1993 and Marcus 2004.)

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A normal form for coverable words

Remark

The word $\mathbf{w}$ is *aba*-coverable iff $\mathbf{w} \in \{ab, aba\}^\omega$.

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Theorem (Mouchard 2000)

Let $q \in \Sigma^*$ and $(r_i), (\ell_i)$ and (b_i) be all the words such that

$$q = \ell_i b_i = b_i r_i.$$

Then $\mathbf{w}$ is q-coverable iff $\mathbf{w} \in \{\ell_1, \ldots, \ell_k\}^\omega$.

Coverability implies “nothing”

Pick your favorite “bad word” $\mathbf{w}$:

- Not uniformly recurrent
- High topological entropy
- No uniform frequencies for factors
- ...
- High Turing degree

and consider its image by the following morphism:

$$a \mapsto ab \quad b \mapsto aba$$

then you get a “bad coverable word”.

(Cf. Marcus, Monteil 2006.)

A stronger coverability notion...

Definition

A word is **multi-scale coverable** if it has infinitely many covers.

Examples:

- Periodic words
- Fixed-points of $a \mapsto ab, b \mapsto aba$ and the like
- Most Sturmian words (Cf. Levé, Richomme 2004.)

... with better dynamical properties

Theorem (Marcus, Monteil 2006)

Let $\mathbf{w}$ be a multi-scale word. Then $\mathbf{w}$ is uniformly recurrent, has uniform factor frequencies and has 0 topological entropy.

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Motivation: connect multi-scale coverability with self-similarity (infinitely many de-substitutions, cf. tilings)

But try to connect these things in $\mathbb{N}$-words first!

Our main tool I

Proposition 1

Let $\mathbf{w} \in \Sigma^{\mathbb{N}}$ and set $p_n = \mathbf{w}[1 \ldots n]$.
Suppose p_i is a cover of $\mathbf{w}$.
Then p_{i+1} is a cover iff p_i is *not* right special.

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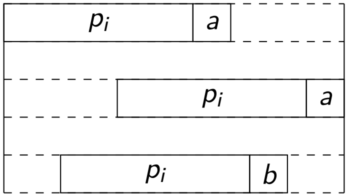
Proof.

If p_i is *not* right special, any occurrence of p_i extends to p_{i+1}.
Conversely, suppose $p_{i+1} = p_i \cdot a$ cover and $p_i \cdot b$ factor of $\mathbf{w}$.



Combinatorial arguments yield $a = b$. □

Our main tool II

Proposition 2

Let $\mathbf{w} \in \Sigma^{\mathbb{N}}$ and set $p_n = \mathbf{w}[1 \ldots n]$.
Suppose p_i is a cover of $\mathbf{w}$. Then p_{i-1} is *not* a cover iff ${p_i}^2$ is a factor of $\mathbf{w}$ and p_{i-1} is not an internal factor of $p_i \cdot p_{i-1}$.

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Proposition 2

Let $\mathbf{w} \in \Sigma^{\mathbb{N}}$ and set $p_n = \mathbf{w}[1 \ldots n]$.
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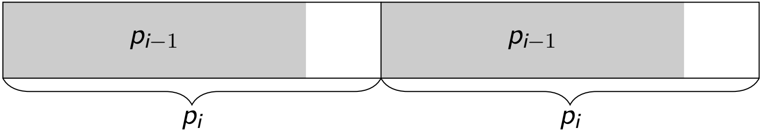
Proof.

Here is the only situation when p_{i-1} is *not* a cover:



where there are no other occurrences of p_{i-1}. □

Recap

Given $\mathbf{w}$ and a cover p_i, we know whether p_{i-1} and p_{i+1} are covers.

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Given $\mathbf{w}$ and a cover p_i, we know whether p_{i-1} and p_{i+1} are covers.

- **Consequence 1:** simpler proof of characterization of covers of the Fibonacci word [1]
- **Consequence 2:** counter-example to show multi-scale $\not\Rightarrow$ self-similar

[1] Christou, Crochemore, Iliopoulos 2002 and Levé, Richomme 2004 and Mousavi, Schaeffer, Shallit 2015.

Multi-scale $\not\Longrightarrow$ self-similar

Remember: any *aba*-coverable word is the image by the morphism

$$a \mapsto ab \quad b \mapsto aba$$

of some other word.
This generalizes to any cover (morphism depending on the cover).

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Wrong! The counter-example is a carefully chosen morphic word.
The proof uses Propositions 1 and 2.

Various (counter-)examples

Proposition (G, R 2015)

1. $\exists$ a multi-scale word s.t. no de-substituted word is coverable.
2. $\exists$ a multi-scale word with no coverable covers.
3. $\exists$ a multi-scale word s.t. the $n+1^{\text{th}}$ cover is coverable by the n^{th} cover.

...

New examples of multi-scale words.
Same techniques for design and proof.

Let's move on to a surprise...

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Theorem

A word $\mathbf{w}$ *is periodic iff*
$\exists n \in \mathbb{N}$ *s.t. all prefixes longer than n are covers.*

Proof.

1. Periodic $\implies$ all prefixes longer than the period are covers
2. Converse: no right special prefixes longer than n

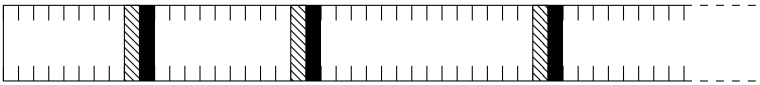
$\square$

(Not the surprise yet)

Question: maximal set of covers?

Idea 1: extension of a right special prefix cannot be a cover.

Idea 2: in aperiodic words, infinitely many prefixes are *not* covers.



▧ = right special ■ = **never** a cover

(We talk about *prefixes*, not letters)

Question: maximal set of covers?

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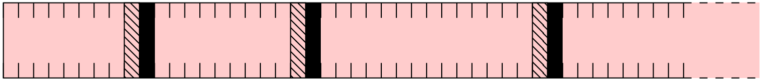

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Could we have this situation?
Could we have an aperiodic word with a "maximal" set of covers?

Answer: yes.

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Surprise!

Theorem (G, R 2016)

*The aperiodic words with a maximal set of covers are **exactly** the standard Sturmian words.*

(A word $\mathbf{w}$ is **standard Sturmian** if it has $n+1$ factors of length n and its prefixes are all left special.)

Thank you!

- Coverable: just a coding
- Multi-scale: good dynamical properties (also in 2D)
- Method to study covers of a given word
- Multi-scale $\not\Rightarrow$ self-similar in 1D
- Other interesting (counter-)examples of multi-scale words
- Covers characterize periodicity
- Covers characterize standard Sturmian words!
- **Perspective:** extension to $\mathbb{Z}$-words and $\mathbb{Z}^2$-words

Thank you for your attention!