Short Flip Sequences to Untangle Segments in the Plane

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───── **Abstract** ─────

A (multi)set of segments in the plane may form a TSP tour, a matching, a tree, or any multigraph. If two segments cross, then we can reduce the total length with the following *flip* operation. We *remove* a pair of crossing segments, and *insert* a pair of non-crossing segments, while keeping the same vertex degrees. The goal of this paper is to devise efficient strategies to flip the segments in order to obtain crossing-free segments after a small number of flips. Linear and near-linear bounds on the number of flips were only known for segments with endpoints in convex position. We generalize these results, proving linear and near-linear bounds for cases with endpoints that are not in convex position. Our results are proved in a general setting that applies to multiple problems, using multigraphs and the distinction between removal and insertion choices when performing a flip.

1 Introduction

The *Euclidean Travelling Salesman Problem (TSP)* is one of the most studied geometric optimization problems. We are given a set P of points in the plane and the goal is to find a tour S of minimum length. While the optimal solution has no crossing segments, essentially all approximation algorithms, heuristics, and PTASs may produce solutions S with crossings. Given S, the only procedure known to obtain a solution S' without crossings and of shorter length is to perform a flip operation. In our case, a *flip* consists of *removing* a pair of crossing segments, and then *inserting* a pair of non-crossing segments preserving a tour (and consequently reducing its length). Flips are performed in *sequence* until a crossing-free tour is obtained, in a procedure called *untangle*.

The same flip operation may be applied in other settings. More precisely, a *flip* consists of removing a pair of crossing segments s_1, s_2 and inserting a pair of segments s_1', s_2' in a way that s_1, s_1', s_2, s_2' forms a cycle and a certain graph *property* is preserved. In the case of TSP tours, the property is being a Hamiltonian cycle. Other properties have also been studied, such as spanning trees, perfect matchings, and multigraphs. Notice that flips preserve the degrees of all vertices and multiple copies of the same edge may appear when we perform a flip on certain graphs.

When the goal is to obtain a crossing-free TSP tour, we are allowed to *choose* which pair of crossing segments to remove in order to perform fewer flips, which we call *removal choice* (Figure 1(a)). Notice that, in a tour, choosing which pair of crossing edges we remove defines which pair of crossing edges we insert. However, this is not the case for matchings and multigraphs. There, we are also allowed to choose which pair of segments to insert among two possibilities, which we call *insertion choice* (Figure 1(b)).
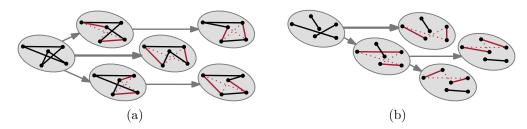
Figure 1 (a) Three untangle sequences for a tour with different *removal choices*. (b) Three untangle sequences for a matching with different *insertion choices*. We highlight the segments removed and inserted at each flip.

Using removal or insertion choices to obtain shorter flip sequences has not been explicitly studied before and opens several new questions, while unifying the solution to multiple reconfiguration problems. Next, we describe previous work according to which choices are used. Throughout, P denotes the set of points and n the number of segments.

Using no choice: Van Leeuwen et al. [27] showed that the *length* (i.e. the number of flips) of any untangle sequence for a TSP tour is $\mathcal{O}(n^3)$ and it is easy to construct $\Omega(n^2)$ examples. The same proof has been rediscovered in the context of matchings [9] after 35 years. If P is in convex position, then the number of crossings decreases at each flip, which gives a tight bound of $\Theta(n^2)$. If all points except the endpoints of t segments are in convex position, then the authors [12] recently showed a bound of $\mathcal{O}(tn^2)$.

Using only insertion choice: Bonnet et al. [9] showed that using only insertion choice, it is possible to untangle a matching using $\mathcal{O}(n^2)$ flips. Let σ be the *spread* of P, that is, the ratio between the maximum and minimum distances among points in P. Using insertion choice, it is also possible to untangle a matching using $\mathcal{O}(n\sigma)$ flips [7].

Using only removal choice: If P is in convex position, then by using $\mathcal{O}(n)$ flips we can untangle a TSP tour [24, 29], as well as a red-blue matching [7], while the best known bound for trees is $\mathcal{O}(n \log n)$ [7]. If instead of convex position, we have colinear red points in a red-blue matching, then $\mathcal{O}(n^2)$ flips suffice [7, 13].

Using both removal and insertion choices: If P is in convex position, then by using $\mathcal{O}(n)$ flips we can untangle a matching [7].

1.1   New Results

Previous results are usually stated for a single graph property. Using choices, we are able to state the results in a more general setting. Proofs that use insertion choice are unlikely to generalize to red-blue matchings, TSP tours, or trees, where insertion choice is not available (still, they may hold for both non-bipartite matchings and multigraphs). In contrast, bounds for multigraphs using only removal choice apply to all these cases. Previously, we only knew linear or near-linear bounds when the points P are in convex position and removal choice is available. The goal of the paper is to obtain linear and near-linear bounds to as many cases as possible, considering near-convex configurations as well as removal and insertion choices.

Let $P = C \cup T$ where C is in convex position and the points of T are outside the convex hull of C, unless otherwise specified. Let S be a multiset of n segments with endpoints P and t be the number of segments with at least one endpoint in T. We prove the following results to untangle S, and some are summarized in Table 1.

Using only insertion choice (Section 2): If $T = \emptyset$, then $\mathcal{O}(n \log n)$ flips suffice. If T is separated from C by two parallel lines, then $\mathcal{O}(tn \log n)$ flips suffice.

Using only removal choice (Section 3): If $|T| \leq 2$ and $t = \mathcal{O}(1)$, then $\mathcal{O}(n \log n)$ flips suffice. In this case, our results hold with the points T being anywhere with respect to the convex hull of C. As the bounds hold for trees, it is useful to compare them against the $\mathcal{O}(n \log n)$ bound for trees from [7] that strongly uses the fact that S forms a tree. The $\mathcal{O}(\log n)$ factor is not present for the special cases of TSP tours and red-blue matchings.

Using both removal and insertion choices (Section 4): If T is separated from C by two parallel lines, then $\mathcal{O}(tn)$ flips suffice. If T is anywhere outside the convex hull of C and S is a matching, then $\mathcal{O}(t^3 n)$ flips suffice.

■ **Table 1** Upper bounds to different versions of the problem with points having $\mathcal{O}(1)$ degree. The letter R corresponds to removal choice, I to insertion choice, and $\emptyset$ to no choice. New results are highlighted in yellow with the theorem number in parenthesis and tight bounds are bold.

Property:	Matching				TSP, Red-Blue	
Choices:	RI	I	R	$\emptyset$	R	$\emptyset$
Convex	$\mathbf{n}$ [7]	$n \log n$ (2.1)	$n \log n$ (3.1)	$\mathbf{n^2}$	$\mathbf{n}$ [7, 24, 29]	$\mathbf{n^2}$
$\|T\| = 1$	$\mathbf{n}$ (4.1)	$n \log n$ (2.2)	$n \log n$ (3.3)	$\mathbf{n^2}$ [12]	$\mathbf{n}$ (3.3)	$\mathbf{n^2}$ [12]
$\|T\| = 2$	$\mathbf{n}$ (4.3)	n^2 [9]	$n \log n$ (3.4)	$\mathbf{n^2}$ [12]	$\mathbf{n}$ (3.4)	$\mathbf{n^2}$ [12]
separated	tn (4.1)	$tn \log n$ (2.2)	tn^2 [12]			
$C \cup T$	$t^3 n$ (4.3)	n^2 [9]	tn^2 [12]			

In a matching or TSP tour, we have $t = \mathcal{O}(|T|)$ and $n = \mathcal{O}(|P|)$, however in a tree, t can be as high as $\mathcal{O}(|T|^2)$. In a multigraph t and n can be much larger than $|T|$ and $|P|$. The theorems describe more precise bounds as functions of all these parameters. For simplicity, the introduction only shows bounds in terms of only n and t.

1.2 Related Reconfiguration Problems

Combinatorial reconfiguration studies the step-by-step transition from one solution to another, for a given combinatorial problem. Many reconfiguration problems are presented in [26]. We give a brief overlook of reconfiguration among line segments using alternative flip operations.

The *2OPT flip* is not restricted to crossing segments. It removes and inserts pairs of segments (the four segments forming a cycle) as the total length decreases. In contrast to flips among crossing segments, the number of 2OPT flips performed may be exponential [14].

It is possible to relax the flip definition even further to all operations that replace two segments by two others forming a cycle [5, 6, 8, 10, 15, 28]. This definition has also been considered for multigraphs [17, 18, 20].

Another type of flip consists of removing a single segment and inserting another one. Such flips are widely studied for triangulations [3, 19, 21, 22, 23, 25]. They have also been considered for non-crossing trees [1] and paths. It is possible to reconfigure any two non-crossing paths if the points are in convex position [4, 11] or if there is *one* point inside the convex hull [2].

1.3 Preliminaries

Throughout, we consider *multigraphs* (P, S) whose vertices P (called *endpoints*) are points in the plane and edges S are a multiset of line *segments*. We assume that the endpoints are in *general position* and that the two endpoints of a segment are distinct. Given two (possibly equal) sets P_1, P_2 of endpoints, we say that a segment is a $P_1 P_2$-*segment* if one endpoint is

in P_1 and the other is in P_2. Similarly, we say that a segment is a P_1-*segment* if at least one endpoint is in P_1.

We say that two segments *cross* if they intersect at a single point that is not an endpoint of either segment. We say that a line *crosses* a segment if they intersect at a single point that is not an endpoint of the segment. We say that a segment or a line h *separates* a set of points P if P can be partitioned into two non-empty sets P_1, P_2 such that every segment p_1p_2 with $p_1 \in P_1, p_2 \in P_2$ crosses h. Given a set of segments S, the *line potential* $\lambda(\ell)$ is the number of segments of S crossed by ℓ. Several proofs in this paper use the following two lemmas from previous papers.

▶ **Lemma 1.1** ([27]). *Given a multiset S of segments and a line ℓ, let $\lambda(\ell)$ be the number of segments in S crossing ℓ. Then, $\lambda(\ell)$ never increases at a flip.*

▶ **Lemma 1.2** ([9]). *Consider a partition $S = \bigcup_i S_i$ of the multiset S of segments and let P_i be the set of endpoints of S_i. If no segment of $\binom{P_i}{2}$ crosses a segment of $\binom{P_j}{2}$ for $i \neq j$, then the sequences of flips in each S_i are independent.*

We say that a segment s is *uncrossable* if for any two endpoints p_1, p_2, we have that p_1p_2 do not cross s. Lemma 1.2 implies that an uncrossable segment cannot be flipped.

Our bounds often have terms like $\mathcal{O}(tn)$ and $\mathcal{O}(n\log|C|)$ that would incorrectly become 0 if t or $\log|C|$ is 0. In order to avoid this problem, factors in the $\mathcal{O}$ notation should be made at least 1. For example, the aforementioned bounds should be respectively interpreted as $\mathcal{O}((1+t)n)$ and $\mathcal{O}(n\log(2+|C|))$.

1.4   Techniques

To prove our results, we combine previous and new *potential functions* with refined strategies and analysis. Van Leeuwen et al. [27] as well as Bonnet et al. [9] consider $\lambda(\ell)$ for a set L of all $\mathcal{O}(|P|^2)$ lines defined by P. Since there always exists a line in L whose potential decreases at a flip, we obtain the $\mathcal{O}(|P|^2 n) = \mathcal{O}(n^3)$ classical bound without any choice. Bonnet et al. [9] show that a set L of $|P| - 1$ parallel lines (with one point between two consecutive lines) suffices with insertion choice. Since there is always an insertion choice that makes some $\lambda(\ell)$ decrease for $\ell \in L$, the $\mathcal{O}(|P|n) = \mathcal{O}(n^2)$ bound follows.

In order to avoid a quadratic dependency in n, new line potentials have to be introduced with careful removal and/or insertion choices. For example, to prove Theorem 4.3, we have to perform several flips in order to find a line ℓ with $\lambda(\ell) = \mathcal{O}(t)$ before applying the line potential argument. In contrast, to prove Theorem 2.2, we modify the line potential to only count the t T-segments. However, with this change the line potential may increase, which we need to handle properly.

Another key potential, inspired by [7] and used for the convex case is the *depth potential* $\delta(p_ap_b)$ of a segment p_ap_b, defined as the number of points between p_a and p_b along the convex hull boundary with a given orientation. Careful removal and insertion choices as well as adaptations of this potential had to be made in order to guarantee that the potential decreases during most flips and never increases by too much. For example, to prove Theorems 2.1 and 2.2, we had to consider the product of the depth, instead of the usual sum. To prove Theorem 3.1, we had to modify the depth potential to only count endpoints of segments that have crossings, which we call the *crossing depth* $\delta_\times(p_ap_b)$.

In the convex case, the number of crossings decreases at each flip, which implies the trivial $\binom{n}{2}$ upper bound. However, the number of crossings may increase when the points are not in convex position. An analysis of the number of crossings is used to bound the number of flips in the proof of Theorem 3.5.

Finally, we use the concept of *splitting* from [9], presented in Lemma 1.2. The difficulty of splitting is to obtain the disjoint sets required by the lemma. For example, in Theorem 4.3, we untangle segments with both endpoints in C last to obtain the desired separation. In Theorem 3.4, we carefully find lines that split the original problem into problems with a smaller value of t that are solved recursively. The special case of uncrossable segments is used in Theorems 3.3 and 3.4.

2 Insertion Choice

In this section, we show how to untangle a multigraph using only insertion choice, that is, our strategies do not choose which pair of crossing segments is removed, but only which pair of segments with the same endpoints is subsequently inserted. We start with the convex case, followed by points outside the convex separated by two parallel lines.

2.1 Convex

Let $P = C = \{p_1, \dots, p_{|C|}\}$ be a set of points in convex position sorted in counterclockwise order along the convex hull boundary (Figure 2(a)). Given a segment $p_a p_b$, we define the *depth* $\delta(p_a p_b) = |b - a|$. This definition resembles but is not the same as the depth used in [7]. We use the depth to prove the following theorem.
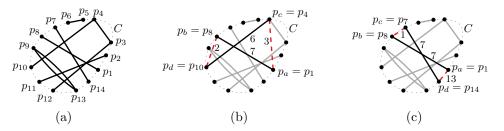

Figure 2 (a) A multigraph (C, S) with $|C| = 14$ points in convex position and $n = 9$ segments. (b) Insertion choice for Case 1 and 2 of the proof of Theorem 2.1. (c) Insertion choice for Case 3.

▶ **Theorem 2.1.** *Every multigraph (C, S) with C in convex position has an untangle sequence of length $\mathcal{O}(n \log |C|) = \mathcal{O}(n \log n)$ using only insertion choice, where $n = |S|$.*

Proof. Let the potential function

$$\phi(S) = \prod_{s \in S} \delta(s).$$

As $\delta(s) \in \{1, \dots, |C| - 1\}$, we have that $\phi(S)$ is integer, positive, and at most $|C|^n$. Next, we show that for any flipped pair of segments $p_a p_b, p_c p_d$ there exists an insertion choice that multiplies $\phi(S)$ by a factor of at most $3/4$, and the theorem follows.

Consider a flip of a segment $p_a p_b$ with a segment $p_c p_d$ and assume without loss of generality that $a < c < b < d$. The contribution of the pair of segments $p_a p_b, p_c p_d$ to the potential $\phi(S)$ is the factor $f = \delta(p_a p_b)\delta(p_c p_d)$. Let f' be the factor corresponding to the pair of inserted segments.

Case 1: If $\delta(p_a p_c) \le \delta(p_c p_b)$, then we insert the segments $p_a p_c$ and $p_b p_d$ and we get $f' = \delta(p_a p_c)\delta(p_b p_d)$ (Figure 2(b)). We notice $\delta(p_a p_b) = \delta(p_a p_c) + \delta(p_c p_b)$. It follows $\delta(p_a p_c) \le \delta(p_a p_b)/2$ and we have $\delta(p_b p_d) \le \delta(p_c p_d)$ and then $f' \le f/2$.

Case 2: If $\delta(p_b p_d) \leq \delta(p_c p_b)$, then we insert the same segments $p_a p_c$ and $p_b p_d$ as previously. We have $\delta(p_a p_c) \leq \delta(p_a p_b)$ and $\delta(p_b p_d) \leq \delta(p_c p_d)/2$, which gives $f' \leq f/2$.

Case 3: If (i) $\delta(p_a p_c) > \delta(p_c p_b)$ and (ii) $\delta(p_b p_d) > \delta(p_c p_b)$, then we insert the segments $p_a p_d$ and $p_c p_b$ (Figure 2(c)). The contribution of the new pair of segments is $f' = \delta(p_a p_d)\delta(p_c p_b)$. We introduce the coefficients $x = \frac{\delta(p_a p_c)}{\delta(p_c p_b)}$ and $y = \frac{\delta(p_b p_d)}{\delta(p_c p_b)}$ so that $\delta(p_a p_c) = x\delta(p_c p_b)$ and $\delta(p_b p_d) = y\delta(p_c p_b)$. It follows that $\delta(p_a p_b) = (1 + x)\delta(p_c p_b)$, $\delta(p_c p_d) = (1 + y)\delta(p_c p_b)$ and $\delta(p_a p_d) = (1 + x + y)\delta(p_c p_b)$. The ratio f'/f is equal to a function $g(x, y) = \frac{1+x+y}{(1+x)(1+y)}$. Due to (i) and (ii), we have that $x \geq 1$ and $y \geq 1$. In other words, we can upper bound the ratio f'/f by the maximum of the function $g(x, y)$ with $x, y \geq 1$. It is easy to show that the function $g(x, y)$ is decreasing with both x and y. Then its maximum is obtained for $x = y = 1$ and it is equal to 3/4, showing that $f' \leq 3f/4$.   ◄

2.2   Separated by Two Parallel Lines

In this section, we prove the following theorem, which is a generalization of Theorem 2.1.
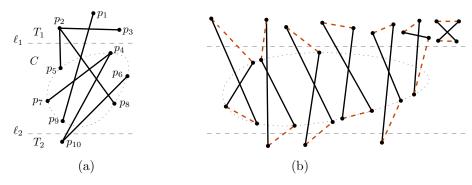

(a)   (b)

Figure 3 (a) Statement of Theorem 2.2. (b) Some insertion choices in the proof of Theorem 2.2.

▶ **Theorem 2.2.** *Consider a multigraph (P, S) with $P = C \cup T_1 \cup T_2$ where C is in convex position and there exist two horizontal lines ℓ_1, ℓ_2, with T_1 above ℓ_1 above C above ℓ_2 above T_2. Let $T = T_1 \cup T_2$, $n = |S|$, and t be the number of T-segments. There exists an untangle sequence of length $\mathcal{O}(t|P|\log|C| + n\log|C|) = \mathcal{O}(tn\log n)$ using only insertion choice.*

Proof. We start by describing the insertion choice for flips involving at least one point in T. Let $p_1, \dots, p_{|P|}$ be the points P sorted vertically from top to bottom. Consider a flip involving the points p_a, p_b, p_c, p_d with $a < b < c < d$. The insertion choice is to create the segments $p_a p_b$ and $p_c p_d$. See Figure 3(b). As in [9], we define the potential η of a segment $p_i p_j$ as

$$\eta(p_i p_j) = |i - j|.$$

Notice that η is an integer between 1 and $|P| - 1$. We define $\eta_T(S)$ as the sum of $\eta(p_i p_j)$ for $p_i p_j \in S$ with p_i or p_j in T. Notice that $0 < \eta_T(S) < t|P|$. It is easy to verify that any flip involving a point in T decreases $\eta_T(S)$ and other flips do not change $\eta_T(S)$. Hence, the number of flips involving at least one point in T is $\mathcal{O}(t|P|)$.

For the flips involving only points of C, we use the same choice as in the proof of Theorem 2.1. The potential function

$$\phi(S) = \prod_{p_i p_j \in S \,:\, p_i \in C \text{ and } p_j \in C} \delta(p_i p_j)$$

is at most $|C|^n$ and decreases by a factor of at most $3/4$ at every flip that involves only points of C.

However, $\phi(S)$ may increase by a factor of $\mathcal{O}(|C|^2)$ when performing a flip that involves a point in T. As such flips only happen $\mathcal{O}(t|P|)$ times, the total increase is at most a factor of $|C|^{\mathcal{O}(t|P|)}$.

Concluding, the number of flips involving only points in C is at most

$$\log_{4/3}\left(|C|^{\mathcal{O}(n)}|C|^{\mathcal{O}(t|P|)}\right) = \mathcal{O}(n\log|C| + t|P|\log|C|). \qquad \blacktriangleleft$$

3 Removal Choice

In this section, we show how to untangle a multigraph using only removal choice. We start with the convex case, followed by 1 point inside or outside the convex, then 2 points outside the convex, 2 points inside the convex, and 1 point inside and 1 outside the convex. As only removal choice is used, all results also apply to red-blue matchings, TSP tours, and trees.

3.1 Convex

Let $P = C = \{p_1, \ldots, p_{|C|}\}$ be a set of points in convex position sorted in counterclockwise order along the convex hull boundary and consider a set of segments S with endpoints P. Given a segment $p_a p_b$ and assuming without loss of generality that $a < b$, we define the *crossing depth* $\delta_\times(p_a p_b)$ as the number of points in $p_{a+1}, \ldots, p_{b-1}$ that are an endpoint of a segment in S that crosses any other segment in S (not necessarily $p_a p_b$). We use the crossing depth to prove the following theorem, which implies a simpler and more general proof of the $\mathcal{O}(n\log n)$ bound for trees [7].
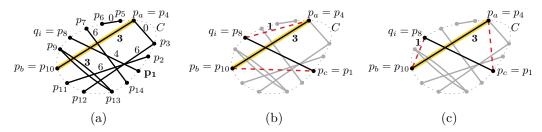

(a)　　　　　　　　　　(b)　　　　　　　　　　(c)

Figure 4 Proof of Theorem 3.1. (a) The segments of a convex multigraph are labeled with the crossing depth. (b,c) Two possible pairs of inserted segments, with one segment of the pair having crossing depth $\lfloor\frac{3}{2}\rfloor = 1$.

▶ **Theorem 3.1.** *Every multigraph (C, S) with C in convex position has an untangle sequence of length $\mathcal{O}(n\log|C|) = \mathcal{O}(n\log n)$ using only removal choice, where $n = |S|$.*

Proof. We repeat the following procedure until there are no more crossings. Let $p_a p_b \in S$ be a segment *with crossings* (hence, crossing depth at least one) and $a < b$ minimizing $\delta_\times(p_a p_b)$ (Figure 4(a)). Let $q_1, \ldots, q_{\delta_\times(p_a p_b)}$ be the points defining $\delta_\times(p_a p_b)$ in order and let $i = \lceil\delta_\times(p_a p_b)/2\rceil$. Since $p_a p_b$ has minimum crossing depth, the point q_i is the endpoint of segment $q_i p_c$ that crosses $p_a p_b$. When flipping $q_i p_c$ and $p_a p_b$, we obtain a segment s (either $s = q_i p_a$ or $s = q_i p_b$) with $\delta_\times(s)$ at most half of the original value of $\delta_\times(p_a p_b)$ (Figure 4(b,c)). Hence, this operation always divides the value of the smallest positive crossing depth by at least two. As the crossing depth is an integer smaller than $|C|$, after performing this operation $\mathcal{O}(\log|C|)$ times, it produces a segment of crossing depth 0. As the segments of crossing depth 0 can no longer participate in a flip, the claimed bound follows. ◀

3.2   One Point Inside or Outside a Convex

In this section, we prove Theorem 3.3. In the case of TSP tours [24, 29] and red-blue matchings [8], the preprocessing to untangle CC-segments takes $\mathcal{O}(n)$ flips. However, in the case of trees [8] and in general (Theorem 3.1), the best bound known is $\mathcal{O}(n \log n)$. We first state a lemma used to prove Theorem 3.3.

▶ **Lemma 3.2.** *Consider a set C of points in convex position, and a multiset S of n crossing-free segments with endpoints in C. Consider the multiset $S \cup \{s\}$ where s is an extra segment with one endpoint in C and one endpoint q anywhere in the plane. There exists an untangle sequence for $S \cup \{s\}$ of length $\mathcal{O}(n)$ using only removal choice.*

Proof. Iteratively flip the segment qp_1 with the segment $p_2p_3 \in S$ crossing qp_1 the farthest from q. This flip inserts a CC-segment p_1p_2, which is impossible to flip again, because the line p_1p_2 is crossing free. The flip does not create any crossing between CC-segments. ◀

We are now ready to state and prove the theorem.

▶ **Theorem 3.3.** *Consider a multigraph (P, S) with $P = C \cup T$ where C is in convex position, and $T = \{q\}$, and such that there is no crossing pair of CC-segments (possibly after a preprocessing for the convex case). Let $n = |S|$ and t be the number of T-segments. There exists an untangle sequence of length $\mathcal{O}(tn)$ using only removal choice.*

Proof. For each segment s with endpoint q with crossing, we apply Lemma 3.2 to s and the CC-segments crossing s. Once a segment s incident to q is crossing free, it is impossible to flip it again as we fall in one of the following cases. Let ℓ be the line containing s.
   Case 1: If ℓ is crossing free, then it splits the multigraph in three partitions: the segments on one side of ℓ, the segments on the other side of ℓ, and the segment s itself.
   Case 2: If ℓ is not crossing free and q is outside the convex hull of C, then s is *uncrossable*.
   Case 3: If q is inside the convex hull of C, then introducing a crossing on s would require that q lies in the interior of the convex quadrilateral whose diagonals are the two segments removed by a flip. The procedure excludes this possibility by ensuring that there are no crossing pair of CC-segments, and, therefore, that one of the removed segment already has q as an endpoint.
   Therefore, we need at most n flips for each of the t segments incident to q. ◀

3.3   Two Points Outside a Convex

In this section, we prove a theorem with a bound that is exponential in t, which makes it of little interest for large t. Notice, however, that in matchings $t \le 2$, in a TSP tour $t \le 4$, and in a binary tree $t \le 6$. Also notice that the definition of t is different from other theorems (here TT-segments are counted twice). Both definitions are equivalent up to a factor of 2, but since t appears in the exponent, they are not exchangeable.

▶ **Theorem 3.4.** *Consider a multigraph (P, S) with $P = C \cup T$ where C is in convex position, the points of T are outside the convex hull of C, and $|T| \le 2$. Let $n = |S|$ and t be the sum of the degrees of the points in T. There exists an untangle sequence of length $\mathcal{O}(2^t d_{conv}(n))$ using only removal choice, where $d_{conv}(n)$ is the number of flips to untangle any multiset of at most n segments with endpoints in convex position.*

Proof. Throughout this proof, we partition the TT-segments respectively the CT-segments into two types: TTI-*segment* and CTI-*segment* if it intersects the interior of the convex hull

of C and *TTO-segment* and *CTO-segment* otherwise. Let $f(t)$ be the number of flips to untangle a multiset S as in the statement of the theorem. The proof proceeds by induction. The base case is $t = 0$, when $f(0) \leq d_{\mathrm{conv}}(n)$ by definition of $d_{\mathrm{conv}}(n)$.

Next, we show how to bound $f(t)$ for $t > 0$, but first we need some definitions. A line ℓ is a *T-splitter* if ℓ is crossing free and either ℓ contains a T-segment or there are T-segments on both sides of ℓ. We abusively say that a segment s is a T-splitter if the line containing s is a T-splitter. A T-splitter is useful because we can apply Lemma 1.2 and solve sub-problems with a lower value of t by induction.

Phase 1: untangle all but one segment by induction. We remove an arbitrary CT-segment or TT-segment s from S. We then use induction to untangle S using $f(t-1)$ flips and insert the segment s back in S afterwards. Notice that all crossings are now on s.

Phase 2.1: apply induction if possible. If S admits a T-splitter ℓ, then we apply Lemma 1.2 to solve each side of ℓ independently using induction.

If S has a crossing-free TTO-segment qq' such that the line qq' is not crossing free, then qq' is uncrossable, and we remove qq' from S and untangle S by induction. Similarly, in the case where $T = \{q, q'\}$ and where qq' is a TTI-segment, if S has a CTO-segment, say pq, then pq is uncrossable, and we remove pq from S and untangle S by induction.

In all the three cases of Phase 2.1 we get $f(t) \leq f(t-1) + f(t_1) + f(t_2)$, where $t_1 + t_2 \leq t$ and $t_1, t_2 \geq 1$.

Phase 2.2: split after one flip. If S contains no T-splitter and if s is a TT-segment, then there remains no CT-segment in S (as every CT-segment shares an endpoint with the TT-segment s that contains all crossings), and s crosses a CC-segment s'. A crossing-free CT-segment would either be a CTI-segment, hence a T-splitter, or a CTO-segment and, hence uncrossable and removed by one of the induction cases of Phase 2.1.

The segment s' becomes a T-splitter after flipping s with s', and we invoke induction. By Lemma 1.2, we get in this case $f(t) \leq f(t-1) + 1 + f(t_1) + f(t_2)$, where $t_1 + t_2 \leq t$ and $t_1, t_2 \geq 1$.

Phase 2.3: split after $\mathcal{O}(n)$ flips. In this case, S contains no T-splitter and s is a CT-segment, say with q as its endpoint in T. While s', the segment of S that crosses s the farthest away from q, is a CC-segment, we flip s and s' and we set s to be the newly inserted CT-segment incident to q. By Lemma 3.2, at most n flips are performed in this loop.

At the end of the loop, either s is crossing free, or s' is a CT-segment, say with q' as its endpoint in T. Then, we also flip s and s'.

Insertion case 1: If two CT-segments are inserted, then, either one of them is uncrossable (this is the case if s' is a CTO-segment), or s' is now a T-splitter (recall that if qq' is a TTI-segment, then all the CTO-segments have been removed at Phase 2.1.).

Insertion case 2: If the TT-segment qq' is inserted, then the inserted CC-segment is crossing free (as in the proof of Lemma 3.2), and, if qq' is not already crossing free, we flip qq' with any segment, say pp'.

Next, we split S as follows. Among the CTI-segments of S which are on the upper (respectively lower) side of the line qq', consider the one whose endpoint p_{upper} (respectively p_{lower}) in C is the closest to the line qq'. The segments of S are either inside or outside the convex quadrilateral $qp_{\mathrm{lower}}q'p_{\mathrm{upper}}$, and we know that only the segments inside may have crossings. By Lemma 1.2, we remove from S all the segments outside $qp_{\mathrm{lower}}q'p_{\mathrm{upper}}$. Recall that, in our case, qq' is a TTI-segment, and all the CTO-segments have been removed at Phase 2.1. The line pp' is finally a T-splitter. Again, by Lemma 1.2, we get in this case $f(t) \leq f(t-1) + n + 2 + f(t_1) + f(t_2)$, where $t_1 + t_2 \leq t$ and $t_1, t_2 \geq 1$.

The last bound on $f(t)$ dominates the recurrence. Using that $f(t_1)+f(t_2) \leq f(t-1)+f(1)$

and $t < n$ we get

$$f(t) \le f(t-1) + n + 2 + f(t_1) + f(t_2) \le \mathcal{O}(n) + 2f(t-1),$$

which solves to $f(t) = \mathcal{O}(2^t d_{\mathrm{conv}}(n))$ as claimed.   ◄

3.4   Two Points inside a Convex

We prove a similar theorem for two points inside the convex hull of C.

▶ **Theorem 3.5.** *Consider a multigraph (P, S) with $P = C \cup T$ where C is in convex position, the points of T are inside the convex hull of C, and $T = \{q, q'\}$. Let $n = |S|$ and t be the number of T-segments. There exists an untangle sequence of length $\mathcal{O}(d_{conv}(n) + tn)$ using only removal choice, where $d_{conv}(n)$ is the number of flips to untangle any multiset of at most n segments with endpoints in convex position.*

Proof. The untangle sequence is decomposed in five phases. At the end of each phase, a new type of crossings is removed, and types of crossings removed in the previous phases are not present, even if they may temporarily appear during the phase.

   Phase 1 (CT × CT). In this phase, we remove all crossings between pairs of CT-segments using $\mathcal{O}(d_{\mathrm{conv}}(t)) = \mathcal{O}(d_{\mathrm{conv}}(n))$ flips. We separately solve two convex sub-problems defined by the CT-segments, one on each side of the line qq'.

   Phase 2 (CC × CC). In this phase, we remove all crossings between pairs of CC-segments using $\mathcal{O}(d_{\mathrm{conv}}(n))$ flips. As no CT-segment has been created, there is still no crossing between a pair of CT segments. Throughout, our removal will preserve the invariant that no pair of CC-segments crosses.

   Phase 3 (CT × non-central CC). We distinguish between a few types of CC-segments. The *central CC*-segments cross the segment qq' (regardless of qq' being in S or not), while the *non-central* do not. The *peripheral CC*-segments cross the line qq' but not the segment qq', while the *outermost CC*-segments do not cross either. In this phase, we remove all crossings between CT-segments and non-central CC-segments.

   Given a non-central CC-segment pp', let the *out-depth* $\delta'(pp')$ be the number of points of C that are contained inside the halfplane bounded by the line pp' and not containing T. Also, let χ be the number of crossings between the non-central CC-segments and the CT-segments. At the end of each step the two following invariants are preserved. (i) No pair of CC-segments crosses. (ii) No pair of CT-segments crosses.

   At each step, we choose to flip the non-central CC-segment pp' of minimum out-depth that crosses a CT-segment. We flip pp' with the CT-segment $q''p''$ (with $q'' \in \{q, q'\}$) that crosses pp' at the point closest to p (Figure 5(a) and Figure 6(a)). One of the possibly inserted pairs may contain a CT-segment s that crosses another CT-segment s', violating the invariant (ii) (Figure 5(b) and Figure 6(b)). If there are multiple such segments s', then we consider s' to be the segment whose crossing with s is closer to q''. We flip s and s' and obtain either two CT-segments (Figure 5(c) and Figure 6(c)) or a CC-segment and the segment qq' (Figure 5(d) and Figure 6(d)). The analysis is divided in two main cases.

   If pp' is an outermost CC-segment (see Figure 5), then case analysis shows that the two invariants are preserved and χ decreases.

   If pp' is a peripheral CC-segment (see Figure 6), then a case analysis shows that the two invariants are preserved and χ has the following behavior. If no CC-segment is inserted, then χ decreases. Otherwise a CC-segment and a TT-segment are inserted and χ may increase by $\mathcal{O}(t)$ (Figure 6(d)). Notice that the number of times the TT-segment qq' is inserted is $\mathcal{O}(t)$, which bounds the total increase by $\mathcal{O}(t^2)$.
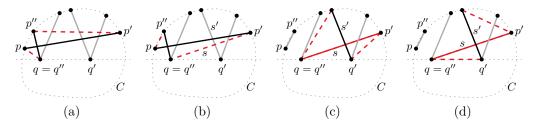
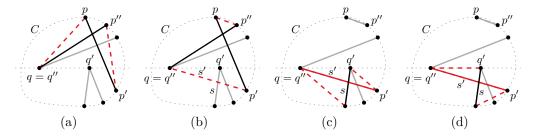**Figure 5** Theorem 3.5, Phase 3 when pp' is an outermost segment.

Figure 6 Theorem 3.5, Phase 3 when pp' is a peripheral segment.

As $\chi = \mathcal{O}(tn)$, the total increase is $\mathcal{O}(t^2)$, and χ decreases at all but $\mathcal{O}(t)$ steps, we have that the number of flips in Phase 3 is $\mathcal{O}(tn)$.

Phase 4 (CT × central CC). At this point, each crossing involves a central CC-segment and either a CT-segment or the TT-segment qq'. In this phase, we remove all crossings between CT-segments and central CC-segments, ignoring the TT-segments. This phase ends with crossings only between qq' and central CC-segments.
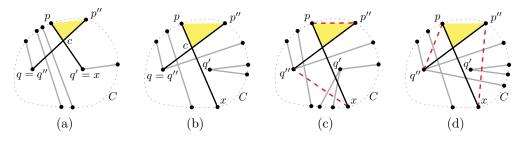

Figure 7 Theorem 3.5, Phase 4. (a) A pair of CT-segments with an ear. (b) A CC-segment and a CT-segment with an ear. (c) Flipping an ear that produces crossing pairs of CT-segments. (d) Flipping an ear that inserts a non-central CC-segment with crossings.

Given four endpoints $q'' \in T$, $p, p'' \in C$, and $x \in C \cup T$, we say that a pair of segments $p''q'', xp \in S$ crossing at a point c contains an *ear* $\widehat{pp''}$ if the interior of the triangle $pp''c$ intersects no segment of S (see Figure 7(a) and 7(b)). Every set of segments with endpoints in $C \cup T$ with $|T| = 2$ that has crossings (not involving the TT-segment) contains an ear (adjacent to the crossing that is farthest from the line qq').

At each *step*, we flip a pair of segments $p''q'', xp$ that contains an ear $\widehat{pp''}$, prioritizing pairs where both segments are CT-segments. Notice that, even though initially we did not have crossing pairs of CT-segments, they may be produced in the flip (Figure 7(c)). If the flip inserts a non-central CC-segment which crosses some CT-segments (Figure 7(d)), then, we perform the following *while loop*. Assume without loss of generality that qq' is horizontal and s is closer to q' than to q. While there exists a non-central CC-segment s with crossings,

we flip s with the CT-segment s' crossing s that comes first according to the following order. As a first criterion, a segment incident to q comes before a segment incident to q'. As a second tie-breaking criterion, a segment whose crossing point with s that is farther from the line qq' comes before one that is closer.

Let $\chi = \mathcal{O}(tn)$ be the number of crossings between central CC-segments and CT-segments plus the number of crossings between CT-segments. A case analysis shows that the value of χ decreases at each step. If no non-central CC-segment is inserted, then the corresponding step consists of a single flip. As χ decreases, there are $\mathcal{O}(tn)$ steps that do not insert a non-central CC-segment.

However, if a non-central CC-segment is inserted, at the end of the step we inserted a CC-segment that can no longer be flipped (Lemma 1.2). As the number of CC-segments is $\mathcal{O}(n)$, we have that the number of times the while loop is executed is $\mathcal{O}(n)$. Since each execution of the while loop performs $\mathcal{O}(t)$ flips, we have a total of $\mathcal{O}(tn)$ flips in this phase.

Phase 5 (TT $\times$ central CC). In this phase, we remove all crossings left, which are between the possibly multiple copies of the TT-segment qq' and central CC-segments. The endpoints of the segments with crossings are in convex position and all other endpoints are outside their convex hull. Hence, by Lemma 1.2, it is possible to obtain a crossing-free multigraph using $\mathcal{O}(d_{\mathrm{conv}}(n))$ flips.   ◀

3.5 One Point inside and One Point Outside a Convex

Given an endpoint p, let $\delta(p)$ denote the degree of p, that is, the number of segments incident to p. The following lemma is used to prove Theorem 3.7.

▶ **Lemma 3.6.** *Consider a multigraph (P, S) with $P = C \cup T$ where C is in convex position, and $T = \{q, q'\}$ such that q is outside the convex hull of C and q' is inside the convex hull of C. Consider that q is the endpoint of a single segment s and all crossings are on s. Let $n = |S|$ and $t = \mathcal{O}(\delta(q'))$ be the number of T-segments. There exists a flip sequence of length $\mathcal{O}(tn)$ using only removal choice that ends with all crossings (if any) on the segment qq'.*

Proof. We proceed as follows, while s has crossings. For induction purpose, let $f(n')$ be the length of the flip sequence in the lemma statement for $n' < n$ segments. Let s' be the segment that crosses s at the point farthest from q. We flip s and s', arriving at one of the three cases below (Figure 8).
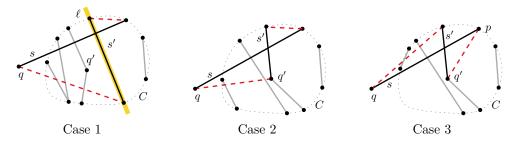

Figure 8 The three cases in the proof of Lemma 3.6.

Case 1 (CT $\times$ CC). In this case, the segment s' is a CC-segment. Notice that the line ℓ containing s' becomes crossing free after the flip. There are segments on both sides of ℓ. If ℓ separates q, q', then we untangle both sides independently (Lemma 1.2) using $\mathcal{O}(n)$ and $\mathcal{O}(tn)$ flips (Theorem 3.3). Otherwise, the segments on one side of ℓ are already crossing free

(because of the specific choice of s') and we inductively untangle the $n' \leq n - 1$ segments on the other side of ℓ using $f(n')$ flips.

Case 2 (CT × CT → CC, TT). If s' is a CT-segment and one of the inserted segments is the TT-segment qq', then the procedure is over as all crossings are on qq'.

Case 3 (CT × CT → CT, CT). In this case two CT-segments are inserted. Let $p \in C$ be an endpoint of $s = qp$. Since the inserted CT-segment $q'p$ is crossing free, Case 3 only happens $\mathcal{O}(t)$ times before we arrive at Case 1 or Case 2.

Putting the three cases together, we obtain the recurrence

$$f(n) \leq \mathcal{O}(t) + f(n'), \text{ with } n' \leq n - 1,$$

which solves to $f(n) = \mathcal{O}(tn)$, as claimed. ◀

We are now ready to prove the theorem.

▶ **Theorem 3.7.** *Consider a multigraph (P, S) with $P = C \cup T$ where C is in convex position, and $T = \{q, q'\}$ such that q is outside the convex hull of C and q' is inside the convex hull of C. Let $n = |S|$ and t be the number of T-segments. There exists an untangle sequence of length $\mathcal{O}(d_{conv}(n) + \delta(q)\delta(q')n) = \mathcal{O}(d_{conv}(n) + t^2 n)$ using only removal choice, where $d_{conv}(n)$ is the number of flips to untangle any multiset of at most n segments with endpoints in convex position.*

Proof. The untangle sequence contains four phases.

Phase 1 (CC × CC). In this phase, we remove all crossings between pairs of CC-segments using $d_{\text{conv}}(n)$ flips. Throughout all the phases, the invariant that no pair of CC-segments crosses is preserved.

Phase 2 (Cq' × CC). In this phase, we remove all crossings between pairs composed of a CC-segment and a CT-segment incident to q' (the point inside the convex hull of C) using $\mathcal{O}(tn)$ flips by Theorem 3.3.

Phase 3 (Cq). At this point, all crossings involve a segment incident to q. In this phase, we deal with all remaining crossings except the crossings involving the segment qq'. Lemma 1.2 allows us to remove the crossings in each CT-segment s incident to q independently, which we do using $\mathcal{O}(\delta(q')n)$ flips using Lemma 3.6. As there are $\delta(q)$ CT-segments adjacent to q, the total number of flips is $\mathcal{O}(\delta(q)\delta(q')n) = \mathcal{O}(t^2 n)$.

Phase 4 (CC × TT). At this point, all crossings involve the TT-segment qq'. The endpoints in C that are adjacent to segments with crossings, together with q', are all in convex position. Hence, the only endpoint not in convex position is q, and we apply Theorem 3.3 using $\mathcal{O}(tn)$ flips.

After the $d_{\text{conv}}(n)$ flips in Phase 1, the number of flips is dominated by Phase 3 with $\mathcal{O}(\delta(q)\delta(q')n) = \mathcal{O}(t^2 n)$ flips. ◀

Notice that, in certain cases (for example in the red-blue case with q, q' having different colors) a flip between two CT-segments never produces two CT-segments. Consequently, Case 3 of the proof of Lemma 3.6 never happens, and the bound in Theorem 3.7 decreases to $\mathcal{O}(d_{\text{conv}}(n) + tn)$.

4 Removal and Insertion Choices

In this section, we show how to untangle a matching or a multigraph using both removal and insertion choices. We start with the case of points outside the convex separated by two parallel lines. Afterwards, we prove an important lemma and apply it to untangle a matching with points outside the convex.

4.1 Separated by Two Parallel Lines

We start with the simpler case in which T is separated from C by two parallel lines. In this case, our bound of $\mathcal{O}(n + t|P|)$ interpolates the tight convex bound of $\mathcal{O}(n)$ from [7] and the $\mathcal{O}(t|P|)$ bound from [9] for t arbitrary segments.

▶ **Theorem 4.1.** *Consider a multigraph (P, S) with $P = C \cup T_1 \cup T_2$ where C is in convex position and there exist two horizontal lines ℓ_1, ℓ_2, with T_1 above ℓ_1 above C above ℓ_2 above T_2. Let $n = |S|$, $T = T_1 \cup T_2$, and t be the number of T-segments. There exists an untangle sequence of length $\mathcal{O}(n + t|P|) = \mathcal{O}(tn)$ using both removal and insertion choices.*

Proof. The algorithm runs in two phases.

Phase 1. We use removal choice to perform the flips involving a point in T. At the end of the first phase, there can only be crossings among segments with all endpoints in C. The insertion choice for the first phase is the following. Let $p_1, \ldots, p_{|P|}$ be the points P sorted vertically from top to bottom. Consider a flip involving the points p_a, p_b, p_c, p_d with $a < b < c < d$. The insertion choice is to create the segments $p_a p_b$ and $p_c p_d$. As in [9], we define the potential η of a segment $p_i p_j$ as $\eta(p_i p_j) = |i - j|$. Notice that η is an integer from 1 to $|P| - 1$. We define $\eta(S)$ as the sum of $\eta(p_i p_j)$ for $p_i p_j \in S$ with p_i or p_j in T. Notice that $0 < \eta(S) < t|P|$. It is easy to verify that any flip involving a point in T decreases $\eta(S)$. Hence, the number of flips in Phase 1 is $\mathcal{O}(t|P|)$.

Phase 2. Since T is outside the convex hull of C, flips between segments with all endpoints in C cannot create crossings with the other segments, which are guaranteed to be crossing free at this point. Hence, it suffices to run an algorithm to untangle a convex set with removal and insertion choice from [7], which performs $\mathcal{O}(n)$ flips. ◀

4.2 Liberating a Line

In this section, we prove the following key lemma, which we use in the following section. The lemma only applies to matchings and it is easy to find a counter-example for multisets (S consisting of n copies of a single segment that crosses pq).

▶ **Lemma 4.2.** *Consider a matching S of n segments with endpoints C in convex position, and a segment pq separating C. Using $\mathcal{O}(n)$ flips with removal and insertion choices on the initial set $S \cup \{pq\}$, we obtain a set of segments that do not cross the line pq.*

Proof. For each flip performed in the subroutine described hereafter, at least one of the inserted segments does not cross the line pq and is removed from S (see Figure 9).
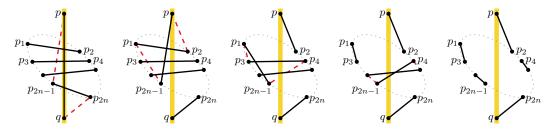

Figure 9 An untangle sequence of the subroutine to liberate the line pq (with $n = 4$).

Preprocessing. First, we remove from S the segments that do not intersect the line pq, as they are irrelevant. Second, anytime two segments in S cross, we flip them choosing to insert the pair of segments not crossing the line pq. One such flip removes two segments

from S. Let p_1p_2 (respectively $p_{2n-1}p_{2n}$) be the segment in S whose intersection point with pq is the closest from p (respectively q). Without loss of generality, assume that the points p_1 and p_{2n-1} are on the same side of the line pq.

First flip. Lemma A.1 applied to the segment pq and the triangle $p_1p_2p_{2n-1}$ shows that at least one of the segments among pp_{2n-1}, qp_1, qp_2 intersects all the segments of S. Without loss of generality, assume that pp_{2n-1} is such a segment, i.e., that pp_{2n-1} crosses all segments of $S \setminus \{p_{2n-1}p_{2n}\}$. We choose to remove the segments pq and $p_{2n-1}p_{2n}$, and we choose to insert the segments pp_{2n-1} and qp_{2n}. As the segment qp_{2n} does not cross the line pq, we remove it from S.

Second flip. We choose to flip the segments pp_{2n-1} and p_1p_2. If n is odd, we choose to insert the pair of segments pp_1, p_2p_{2n-1}. If n is even, we insert the segments pp_2, p_1p_{2n-1}.

By convexity, one of the inserted segment (the one with endpoints in C) crosses all other $n-2$ segments. The other inserted segment (the one with p as one of its endpoints) does not cross the line pq, so we remove it from S. Note that the condition on the parity of n is there only to ensure that the last segment $p_{2n-3}p_{2n-2}$ is dealt with at the last flip.

Remaining flips. We describe the third flip. The remaining flips are performed similarly. Let s be the previously inserted segment. Let p_3p_4 be the segment in S whose intersection point with pq is the closest from p. Without loss of generality, assume that p_3 is on the same side of the line pq as p_1 and p_{2n-1}.

We choose to flip s with p_3p_4. If $s = p_2p_{2n-1}$, we choose to insert the pair of segments p_2p_4, p_3p_{2n-1}. If $s = p_1p_{2n-1}$, we choose to insert the pair of segments p_1p_3, p_4p_{2n-1}.

By convexity, one inserted segment (the one with p_{2n-1} as an endpoint) crosses all other $n-3$ segments. The other inserted segment does not cross the line pq, so we remove it from S. Note that the insertion choice described is the only viable one, as the alternative would insert a crossing-free segment crossing the line pq that cannot be removed. ◀

4.3 Points Outside a Convex

We are now ready to prove the following theorem, which only applies to matchings because it uses Lemma 4.2.

▶ **Theorem 4.3.** *Consider a matching S consisting of n segments with endpoints $P = C \cup T$ where C is in convex position and T is outside the convex hull of C. Let $t = |T|$. There exists an untangle sequence of length $\mathcal{O}(t^3n)$ using both removal and insertion choices.*

Proof. Throughout this proof, we partition the TT-segments into two types: *TTI-segment* if it intersects the interior of the convex hull of C and *TTO-segment* otherwise.

TT-segments. At any time during the untangle procedure, if there is a TTI-segment s that crosses more than t segments, we apply Lemma 4.2 to liberate s from every CC-segment using $\mathcal{O}(n)$ flips. Let ℓ be the line containing s. Since $\lambda(\ell)$ cannot increase (Lemma 1.1), $\lambda(\ell) < t$ after Lemma 4.2, and there are $\mathcal{O}(t^2)$ different TTI-segments, it follows that Lemma 4.2 is applied $\mathcal{O}(t^2)$ times, performing a total $\mathcal{O}(t^2n)$ flips. As the number of times s is inserted and removed differ by at most 1 and $\lambda(\ell)$ decreases at each flip that removes s, it follows that s participates in $\mathcal{O}(t)$ flips. As there are $\mathcal{O}(t^2)$ different TTI-segments, the total number of flips involving TTI-segments is $\mathcal{O}(t^3)$.

We define a set L of $\mathcal{O}(t)$ lines as follows. For each point $q \in T$, we have two lines $\ell_1, \ell_2 \in L$ that are the two tangents of the convex hull of C that pass through q. As the lines $\ell \in L$ do not separate C, the potential $\lambda(\ell) = \mathcal{O}(t)$. When flipping a TTO-segment q_1q_2 with another segment q_3p with $q_3 \in T$ (p may be in T or in C), we make the insertion choice of creating a TTO-segment q_1q_3 such that there exists a line $\ell \in L$ whose potential

$\lambda(\ell)$ decreases. It is easy to verify that ℓ always exist (see Lemmas B.1 and B.2 in the Appendix). Hence, the number of flips involving TTO-segments is $\mathcal{O}(t^2)$ and the number of flips involving TT-segments in general is $\mathcal{O}(t^3)$.

All except pairs of CC-segments. We keep flipping segments that are not both CC-segments with the following insertion choices. Whenever we flip two CT-segments, we make the insertion choice of creating a TT-segment. Hence, as the number of flips involving TT-segments is $\mathcal{O}(t^3)$, so is the number of flips of two CT-segments.

Whenever we flip a CT-segment p_1q with $q \in T$ and a CC-segment p_3p4, we make the following insertion choice. Let $v(q)$ be a vector such that the dot product $v(q) \cdot q < v(q) \cdot p$ for all $p \in C$, that is, v is orthogonal to a line ℓ separating q from C and pointing towards C. We define the potential $\eta(p_xq)$ of a segment with $p_x \in C$ and $q \in T$ as the number of points $p \in C$ such that $v(q) \cdot p < v(q) \cdot p_x$, that is the number of points in C before p_x in direction v. We choose to insert the segment p_xq that minimizes $\eta(p_xq)$ for $x = \{1, 2\}$. Let $\eta(S)$ be the sum of $\eta(p_xq)$ for all CT-segments p_xq in S. It is easy to see that $\eta(S)$ is $\mathcal{O}(t|C|)$ and decreases at each flip involving a CT-segment (not counting the flips inside Lemma 4.2).

There are two situation in which $\eta(S)$ may increase. One is when Lemma 4.2 is applied, which happens $\mathcal{O}(t^2)$ times. Another one is when a TT-segment and a CC-segment flip, creating two CT-segments, which happens $\mathcal{O}(t^3)$ times. At each of these two situations, $\eta(S)$ increases by $\mathcal{O}(|C|)$. Consequently, the number of flips between a CT-segment and a CC-segment is $\mathcal{O}(t^3|C|) = \mathcal{O}(t^3n)$.

CC-segments. By removal choice, we choose to flip the pairs of CC-segments last (except for the ones flipped in Lemma 4.2). As T is outside the convex hull of C, flipping two CC-segments does not create crossings with other segments (Lemma 1.2). Hence, we apply the algorithm from [7] to untangle the remaining segments using $\mathcal{O}(n)$ flips.   ◀

---- **References** ----

1 Oswin Aichholzer, Brad Ballinger, Therese Biedl, Mirela Damian, Erik D Demaine, Matias Korman, Anna Lubiw, Jayson Lynch, Josef Tkadlec, and Yushi Uno. Reconfiguration of non-crossing spanning trees. *arXiv preprint*, 2022. URL: https://arxiv.org/abs/2206.03879.

2 Oswin Aichholzer, Kristin Knorr, Maarten Löffler, Zuzana Masárová, Wolfgang Mulzer, Johannes Obenaus, Rosna Paul, and Birgit Vogtenhuber. Flipping plane spanning paths. In *International Conference and Workshops on Algorithms and Computation (WALCOM)*, 2023. URL: https://arxiv.org/abs/2202.10831, doi:10.1007/978-3-031-27051-2_5.

3 Oswin Aichholzer, Wolfgang Mulzer, and Alexander Pilz. Flip distance between triangulations of a simple polygon is NP-complete. *Discrete & Computational Geometry*, 54(2):368–389, 2015. doi:10.1007/s00454-015-9709-7.

4 Selim G Akl, Md Kamrul Islam, and Henk Meijer. On planar path transformation. *Information processing letters*, 104(2):59–64, 2007. doi:10.1016/j.ipl.2007.05.009.

5 Sergey Bereg and Hiro Ito. Transforming graphs with the same degree sequence. In *Computational Geometry and Graph Theory*, pages 25–32, 2008. doi:10.1007/978-3-540-89550-3_3.

6 Sergey Bereg and Hiro Ito. Transforming graphs with the same graphic sequence. *Journal of Information Processing*, 25:627–633, 2017. doi:10.2197/ipsjjip.25.627.

7 Ahmad Biniaz, Anil Maheshwari, and Michiel Smid. Flip distance to some plane configurations. *Computational Geometry*, 81:12–21, 2019. URL: https://arxiv.org/abs/1905.00791, doi:10.1016/j.comgeo.2019.01.008.

8 Marthe Bonamy, Nicolas Bousquet, Marc Heinrich, Takehiro Ito, Yusuke Kobayashi, Arnaud Mary, Moritz Mühlenthaler, and Kunihiro Wasa. The perfect matching reconfiguration problem. In *44th International Symposium on Mathematical Foundations of Computer Science (MFCS)*, volume 138 of *LIPIcs*, pages 80:1–80:14, 2019. doi:10.4230/LIPIcs.MFCS.2019.80.

9   Édouard Bonnet and Tillmann Miltzow. Flip distance to a non-crossing perfect matching. *arXiv*, 1601.05989, 2016. URL: http://arxiv.org/abs/1601.05989.

10 Nicolas Bousquet and Alice Joffard. Approximating shortest connected graph transformation for trees. In *Theory and Practice of Computer Science*, pages 76–87, 2020. doi:10.1007/978-3-030-38919-2_7.

11 Jou-Ming Chang and Ro-Yu Wu. On the diameter of geometric path graphs of points in convex position. *Information processing letters*, 109(8):409–413, 2009. doi:10.1016/j.ipl.2008.12.017.

12 Guilherme D. da Fonseca, Yan Gerard, and Bastien Rivier. On the longest flip sequence to untangle segments in the plane. In *International Conference and Workshops on Algorithms and Computation (WALCOM)*, volume to appear of *Lecture Notes in Computer Science*, 2023. URL: https://arxiv.org/abs/2210.12036, doi:10.1007/978-3-031-27051-2_10.

13 Arun Kumar Das, Sandip Das, Guilherme D. da Fonseca, Yan Gerard, and Bastien Rivier. Complexity results on untangling red-blue matchings. *Computational Geometry*, 111:101974, 2023. URL: https://arxiv.org/abs/2202.11857, doi:10.1016/j.comgeo.2022.101974.

14 Matthias Englert, Heiko Röglin, and Berthold Vöcking. Worst case and probabilistic analysis of the 2-Opt algorithm for the TSP. *Algorithmica*, 68(1):190–264, 2014. doi:10.1007/s00453-013-9801-4.

15 Péter L Erdős, Zoltán Király, and István Miklós. On the swap-distances of different realizations of a graphical degree sequence. *Combinatorics, Probability and Computing*, 22(3):366–383, 2013. doi:10.1017/S0963548313000096.

16 Branko Grünbaum. Polygons in arrangements generated by n points. *Mathematics Magazine*, 46(3):113–119, 1973. doi:10.1080/0025570X.1973.11976293.

17 Seifollah Louis Hakimi. On realizability of a set of integers as degrees of the vertices of a linear graph. I. *Journal of the Society for Industrial and Applied Mathematics*, 10(3):496–506, 1962.

18 Seifollah Louis Hakimi. On realizability of a set of integers as degrees of the vertices of a linear graph II. uniqueness. *Journal of the Society for Industrial and Applied Mathematics*, 11(1):135–147, 1963.

19 Ferran Hurtado, Marc Noy, and Jorge Urrutia. Flipping edges in triangulations. *Discrete & Computational Geometry*, 22(3):333–346, 1999.

20 Alice Joffard. *Graph domination and reconfiguration problems*. PhD thesis, Université Claude Bernard Lyon 1, 2020.

21 Charles L Lawson. Transforming triangulations. *Discrete Mathematics*, 3(4):365–372, 1972.

22 Anna Lubiw and Vinayak Pathak. Flip distance between two triangulations of a point set is NP-complete. *Computational Geometry*, 49:17–23, 2015. doi:10.1016/j.comgeo.2014.11.001.

23 Naomi Nishimura. Introduction to reconfiguration. *Algorithms*, 11(4), 2018. doi:10.3390/a11040052.

24 Yoshiaki Oda and Mamoru Watanabe. The number of flips required to obtain non-crossing convex cycles. In *Kyoto International Conference on Computational Geometry and Graph Theory*, pages 155–165, 2007.

25 Alexander Pilz. Flip distance between triangulations of a planar point set is APX-hard. *Computational Geometry*, 47(5):589–604, 2014. doi:10.1016/j.comgeo.2014.01.001.

26 Jan van den Heuvel. The complexity of change. *Surveys in Combinatorics*, 409:127–160, 2013.

27 Jan van Leeuwen and Anneke A. Schoone. Untangling a traveling salesman tour in the plane. In *7th Workshop on Graph-Theoretic Concepts in Computer Science*, 1981.

28 Todd G Will. Switching distance between graphs with the same degrees. *SIAM Journal on Discrete Mathematics*, 12(3):298–306, 1999. doi:10.1137/S0895480197331156.

29 Ro-Yu Wu, Jou-Ming Chang, and Jia-Huei Lin. On the maximum switching number to obtain non-crossing convex cycles. In *26th Workshop on Combinatorial Mathematics and Computation Theory*, pages 266–273, 2009.

A   Auxiliary Lemma of Section 4.2

In this section, we prove Lemma A.1 used in the proof of Lemma 4.2.

Recall that, in the proof of Lemma 4.2, we have a convex quadrilateral $p_1p_2p_{2n}p_{2n-1}$ and a segment pq crossing the segments p_1p_2 and $p_{2n}p_{2n-1}$ in this order when drawn from p to q, and we invoke Lemma A.1 to show that at least one of the segments among pp_{2n-1}, qp_1, qp_2 intersects all the segments of S. Before proving Lemma A.1, we detail how to apply it to this context.

Lemma A.1 applied to the segment pq and the triangle $p_1p_2p_{2n-1}$ asserts that at least one of the following pairs of segments cross: pp_{2n-1}, p_1p_2, or qp_1, p_2p_{2n-1}, or qp_2, p_1p_{2n-1}. If the segments pp_{2n-1}, p_1p_2 cross, then we are done. If the segments qp_1, p_2p_{2n-1} cross, then the segments $qp_1, p_{2n}p_{2n-1}$ also cross and we are done. If the segments qp_2, p_1p_{2n-1} cross, then the segments $qp_2, p_{2n}p_{2n-1}$ also cross and we are done.

Next, we state and prove Lemma A.1.

▶ **Lemma A.1.** *For any triangle abc, for any segment pq intersecting the interior of the triangle abc, there exists a segment $s \in \{pa, pb, pc, qa, qb, qc\}$ that intersects the interior of the triangle abc.*

Proof. If all a, b, c, p, q are in convex position, then p and the point among a, b, c that is not adjacent to p on the convex hull boundary define the segment s. Otherwise, since p, q are not adjacent on the convex hull boundary, assume without loss of generality that a is not a convex hull vertex and p, b, q, c are the convex hull vertices in order. Then, either ap or aq intersects bc.   ◀

B   Auxiliary Lemmas of Section 4.3

In this section, we prove Lemma B.2 and Lemma B.1 used in the proof of Theorem 4.3.

Recall that, in the proof of Theorem 4.3, we define a set L of lines as follows. For each point $q \in T$, we have two lines $\ell_1, \ell_2 \in L$ that are the two tangents of the convex hull of C that pass through q. When flipping a TTO-segment q_1q_2 with another segment q_3p with $q_3 \in T$ (p may be in T or in C), we make the insertion choice of creating a TTO-segment q_1q_3 such that there exists a line $\ell \in L$ whose potential $\lambda(\ell)$ decreases. We invoke Lemma B.1 and Lemma B.2 to show that such a line ℓ always exist.

Indeed, by Lemma B.1, it is enough to show that there exists a line $\ell \in L$ containing one of the points q_1, q_2, q_3 that crosses one of the segments q_1q_2 or q_3p. This is precisely what Lemma B.2 shows.

Next, we state prove Lemma B.1 and Lemma B.2.

▶ **Lemma B.1.** *Consider two crossing segments p_1p_2, p_3p_4 and a line ℓ containing p_1 and crossing p_3p_4. Then, one of the two pairs of segments p_1p_3, p_2p_4 or p_1p_4, p_2p_3 does not cross ℓ. In other words, there exists an insertion choice to flip p_1p_2, p_3p_4 such that the number of segments crossing ℓ decreases.*

Proof. Straightforward.   ◀

▶ **Lemma B.2.** *Consider a closed convex body B and two crossing segments q_1q_3, q_2q_4 whose endpoints q_1, q_2, q_3 are not in B, and whose endpoint q_4 is not in the interior of B. If the segment q_1q_3 does not intersect the interior of B, then at least one of the six lines tangent to B and containing one of the endpoints q_1, q_2, q_3 is crossing one of the segments q_1q_3, q_2q_4. (General position is assumed, meaning that the aforementioned six lines are distinct, i.e., each line does not contain two of the points q_1, q_2, q_3, q_4.)*

(a)   (b)
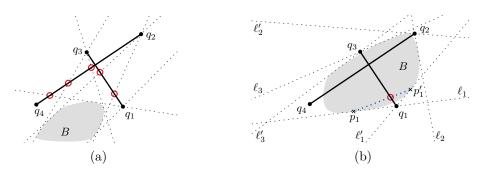
Figure 10 (a) In the statement of Lemma B.2, we assert the existence of points, circled in the figure, which are the intersection of a line tangent to B and containing one of the points q_1, q_2, q_3. (b) In the proof of Lemma B.2 by contraposition, we exhibit a point, circled in the figure, showing that B intersects one of the segment q_1q_3.

Proof. For all $i \in \{1, 2, 3\}$, let ℓ_i and ℓ'_i be the two lines containing q_i and tangent to B. By contraposition, we assume that none of the six lines $\ell_1, \ell'_1, \ell_2, \ell'_2, \ell_3, \ell'_3$ crosses one of the segments q_1q_3, q_2q_4. In other words, we assume that the six lines are tangent to the convex quadrilateral $q_1q_2q_3q_4$. It is well known that, if $m \geq 5$, then any arrangement of m lines or more admits at most one face with m edges (see [16] for example). Therefore, B is contained in the same face of the arrangement of the six lines as the quadrilateral $q_1q_2q_3q_4$. Let p_1 (respectively p'_1) be a contact point between the line ℓ_1 (respectively ℓ'_1) and the convex body B. The segment $p_1p'_1$ crosses the segment q_1q_3 and is contained in B by convexity, concluding the proof by contraposition.   ◄